Optical diagnostics in fluid mechanics

Metrology of particles

Part 1 : Fundamentals

Kuan Fang REN
Email: fang.ren@coria.fr
Tel: 02 32 95 37 43

UMR 6614/ CORIA
CNRS - INSA & University of Rouen
Outline of course

- Introduction
- Fundamentals
- Macroscopic models of light scattering by particles
  - Approximate models
  - Rigorous theories
  - Numerical methods
- Measurement techniques
  - LDV, PDA
  - Imaging techniques
  - Refractometry
  - Extinction spectrometry
- Research topics

- Elastic scattering
- Non-elastic scattering

- Simple scattering
  - Multiple scattering
  - Cohérente scattering
Introduction – macroscopic models

- **Approximate models**
  - Rayleigh: \( l << \lambda \)
  - Rayleigh-Gans: \(|m - 1| << 1\)
  - Diffraction: \( l \sim \lambda \)
  - GTD - Geometrical Theory of Diffraction,
  - Geometrical optics: \( l >> \lambda \)
  - VCRM – Vectorial Complex Ray Model

- **Rigorous theories**
  - TLM – Lorenz-Mie Theory
  - GLMT – Generalized Lorenz-Mie Theory
  - Debye theory / series

- **Numerical methods**
  - T-Matrices,
  - DDA - Dipole discretization Approximation (ADDA, DDSCAT)
  - FDTD - Finite-difference time-domain
  - MoM – Method of Moment
  - FEM – Finite Element Method
Introduction – Measurement techniques

- **ADL – PDA:**
  - velocity, size
  - individual particle

- **Imaging techniques (holography, PIV, HPIV, ...):**
  - Size, velocity
  - individual particle or cloud

- **Rainbow refractometry:**
  - Size, refractive index (very accurate)
  - Big particles

- **Extinction spectrometry**
  - Size, concentration
  - Small particles (d~λ)
Electromagnetic wave (EM) and its properties

Two polarisations:

\[
\begin{align*}
E_x &= A_x \cos(\omega t - k \cdot r - \phi_0) \\
E_y &= A_y \cos(\omega t - k \cdot r - \phi_0)
\end{align*}
\]

\[
E = E_0 e^{i(\omega t - \vec{k} \cdot \vec{r} - \phi_0)}
\]

(complex fonction)

In a homogeneous and isotropic medium:

\[
D = \varepsilon E, \quad B = \mu H
\]

\[
H = \frac{1}{\mu \omega} k \times E
\]

\(E\) – electric field

\(H\) – magnetic field

\(\varepsilon\) – permittivity

\(\mu\) – permeability
Fundamentals – complex refractive index

Complex refractive index: 
\[ \tilde{m}^2 = \varepsilon \]
\[ \tilde{m} = m_r - m_i i \]

Real part – velocity: 
\[ m_r = \frac{c}{v} \]

Examples:
In the vacuum (air): 
\[ c = 3 \times 10^8 \text{ m}\cdot\text{s}^{-1}, \quad \lambda = 0.6328 \mu\text{m} \]
\[ n_{\text{water}} = 1.33 \quad \nu_{\text{water}} = 2.26 \times 10^8 \text{ m}\cdot\text{s}^{-1}, \quad \lambda_{\text{water}} = 0.4758 \mu\text{m} \]
\[ n_{\text{glass}} = 1.5 \quad \nu_{\text{glass}} = 2.00 \times 10^8 \text{ m}\cdot\text{s}^{-1}, \quad \lambda_{\text{glass}} = 0.4219 \mu\text{m} \]

- \( n > n' \): \( v < v' \), \( \lambda < \lambda' \)
- \( n < n' \): \( v > v' \), \( \lambda > \lambda' \)
- \( n = n' \): \( v = v' \), \( \lambda = \lambda' \)
Fundamentals – complex refractive index

Definition and physical interpretation of refractive index

imaginary part - absorption:

Penetration depth:

\[ d = \frac{1}{m_i k_0} = 0.16 \frac{\lambda}{m_i} \]

\[ E = E_0 e^{i(\omega t - m_i k_0 z + \phi)} \]
\[ = E_0 e^{i\omega t - im_i k_0 z - m_i k_0 z + i\phi} \]
\[ = E_0 e^{-m_i k_0 z} e^{i(\omega t - m_i k_0 z + \phi)} \]

Amplitude à z:
\[ E_0(z) = E_0(z = 0) e^{-m_i k_0 z} \]

Penetration depth \( d \):
\[ \frac{E_0(z = d)}{E_0(z = 0)} = e^{-1} \]
\[ \Rightarrow d = \frac{1}{m_i k_0} = 0.16 \frac{\lambda}{m_i} \]

\( \lambda = 0.6328 \mu\text{m} \)

\( m_i = 0.1, \ d = 1 \mu\text{m} \)

\( m_i = 0.0001, \ d = 1 \text{mm} \)
Fundamentals – balance of energy

Energy density and light intensity

Energy density (J/m³): \[ u = \frac{1}{2} \left( \vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right) \]

Poynting vector (W/m²):
\[
\vec{S} = \vec{E} \wedge \vec{H} = \frac{1}{2} \text{Re} \left( \vec{E} \wedge \vec{H}^* \right)
\]

Complex function

In an isotropic medium:
\[ \vec{S} = \nu u \vec{n} \]

Intensity:
\[ I = \| \vec{S} \| \propto E^2 \]
Fundamentals – Polarization

Relation between incident wave and scattered wave

In far field

\[
\begin{pmatrix}
E_{\|s} \\
E_{\perp s}
\end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix}
S_2 & S_3 \\
S_4 & S_1
\end{pmatrix} \begin{pmatrix}
E_{\|i} \\
E_{\perp i}
\end{pmatrix}
\]

\[S_3 = S_4 = 0\] for homogeneous or multilayered sphere

Diagram of a sphere illuminated by a Gaussian beam (off axis), perpendicular polarization in red and parallel in green
Fundamentals – scattering diagram

Angular distribution of intensity of the scattered wave in far field

A sphere of diameter \( d = \alpha \lambda / \pi \) and refractive index \( m = 1.33 \) illuminated by a plane wave

\[
I(r, \theta, \varphi) = \frac{I_0 F(\theta, \varphi)}{k^2 r^2}
\]

\[
I_\parallel(\theta) = F(\theta, \phi = 0) = |S_2|^2
\]

\[
I_\perp(\theta) = F(\theta, \phi = 90^\circ) = |S_1|^2
\]

Size parameter :

\[
\alpha = \frac{\pi d}{\lambda}
\]
Fundamentals – Extinction, scattering, absorption

Integral properties of a scattering particle

Outside of the particle:

**Poynting vector:**

\[
S = \frac{1}{2} \text{Re}\{E \times H^*\} = S_i + S_s + S_{ext}
\]

\[
S_i = \frac{1}{2} \text{Re}\{E_i \times H_i^*\}
\]

\[
S_s = \frac{1}{2} \text{Re}\{E_s \times H_s^*\}
\]

\[
S_{ext} = \frac{1}{2} \text{Re}\{E_i \times H_s^* + E_s \times H_i^*\}
\]

**Energy balance**

\[
W_a = -\int_A S \cdot e_r dA = W_i - W_s + W_{ext}
\]

\[
W_i = -\int_A S_i \cdot e_r dA, \quad W_s = \int_A S_s \cdot e_r dA, \quad W_{ext} = -\int_A S_{ext} \cdot e_r dA
\]

\[
W_{ext} = W_{abs} + W_{sca}
\]
Fundamentals – Extinction, scattering, absorption

Definition of sections and factors

Efficiency sections:

Absorption sections:
\[ C_{abs} = \frac{W_{abs}}{I_i} \]

Scattering section:
\[ C_{sca} = \frac{W_{sca}}{I_i} \]

Extinction section:
\[ C_{ext} = \frac{W_{ext}}{I_i} \]

Physical interpretation

Absorption
\[ + \]
Scattering
\[ \parallel \]
total perturbation

\[ A \] is the geometrical section of the particle area projected in the plan \( \perp \) incident direction.

Efficiency factors

\[ Q_{ext} = \frac{C_{ext}}{A}, \quad Q_{abs} = \frac{C_{abs}}{A}, \quad Q_{sca} = \frac{C_{sca}}{A} \]

Transparent /no absorbing

\[ C_{abs} = 0, \quad C_{ext} = C_{sca} \]

\[ Q_{abs} = 0, \quad Q_{ext} = Q_{sca} \]
Fundamentals – Extinction, scattering, absorption

Why is the sky blue and the sun red at the rising and at the set?

Small particle

\[ Q_{\text{ext}} \sim \frac{d^4}{\lambda^4} \]

Big particle

\[ Q_{\text{ext}} \to 2 \]

Read and understand the graphics

\[ Q_{\text{ext}}(\alpha,m) = Q_{\text{ext}}(\lambda,\alpha,m) \]

Transparent particle

\[ Q_{\text{ext}}(\lambda,\alpha,m) = Q_{\text{ext}}(\lambda,\alpha,m) \]
The greater the absorption, the more smooth is the curve.
The higher the absorption, the smaller the extinction factor.
For small particles: \( Q_{\text{ext}} \propto \alpha \)
For big particles: \( Q_{\text{ext}} \to 1 \)
i.e.: \( C_{\text{ext}} \to A \)
Fundamentals – Gaussian beam

Intensity distribution and geometric shape

(a). *Intensity* decreases exponentially as function of $r^2$, it also evolves along the beam axis.

$$I(r, z) = I_0 \left[ \frac{w_0}{w(z)} \right]^2 \exp \left[ -\frac{2r^2}{w^2(z)} \right]$$

(b). Beam waist radius

$$I(r = w(z)) = \frac{I(r = 0, z)}{e^2}$$

$$w(z) = w_0 \sqrt{1 + \left( \frac{\lambda z}{\pi w_0^2} \right)^2}$$

(c). Divergence of the beam

$$\theta_0 = \arctan \left( \frac{\lambda}{\pi w_0} \right)$$

(d). Rayleigh length: $z_R = \frac{\pi w_0^2}{\lambda}$.

$$I(0, z_R) = \frac{I_0}{2}, \quad w(z_R) = \sqrt{2}w_0$$
Fundamentals – Gaussian beam

Some examples

Divergence:
1. \( \lambda = 600 \text{ nm} \):
   \( w_0 = 10 \mu \text{m}: z_R = 500 \mu \text{m}, \theta = 0.02 \text{ rad} \)
   \( w_0 = 1 \text{ mm}: z_R = 5 \text{ m}, \theta = 0.0002 \text{ rad} \)
   \( w_0 = 1 \text{ cm}: z_R = 500 \text{ m}, \theta = 0.00002 \text{ rad} \)

2. \( w_0 = 100 \mu \text{m} \):
   \( \lambda = 10.6 \mu \text{m (CO2)}: z_R = 3 \text{ mm}, \theta = 0.034 \text{ rad} \)
   \( \lambda = 0.6328 \mu \text{m (He-Ne)}: z_R = 5 \text{ cm}, \theta = 0.002 \text{ rad} \)
   \( \lambda = 0.488 \mu \text{m (bleu YAG)}: z_R = 6.4 \text{ cm}, \theta = 1.5 \text{ mrad} \)

Intensity:

At the center:
\[ I = I_0 \]

On the border of the beam:
\[ I(r = w_0) = \frac{I(r = 0, z = 0)}{e^2} = \frac{I_0}{e^2} \]

Total power of the beam:
\[
P = \int I(r, z) dS = \int_0^{2\pi} \int_0^\infty I(r, z) r dr d\theta
= 2\pi I_0 \left[ \frac{w_0}{w(z)} \right]^2 \int_0^\infty \exp \left[ -\frac{2r^2}{w^2(z)} \right] rdr
= I_0 \frac{\pi w_0^2}{2}
\]
Thin lens equation
Position of the waists before and after a lens

\[
\frac{1}{s'} - \frac{1}{s + \frac{z_R^2}{s + f'}} = \frac{1}{f'}
\]
or

\[
s' = f' \left[1 - \frac{f'(s + f')}{(s + f')^2 + z_R^2}\right]
\]
Consequences:

• For quasi-parallel beams, $z_R$ is much greater than all other distances, therefore $s' = f'$, behaviour of the geometrical optics.

• If the beam waist is in the object focal plane of a lens, the beam waist is at the image focal plane of the lens, since $s = -f \Rightarrow s' = f'$. This result is radically different from the geometrical optics.

• The conjugate relation between $s$ and $s'$ depends not only on $f$ but also on $z_R$, hence on $\lambda$ and $w_0$, which is different from the geometrical optics.
**Fundamentals – Gaussian beam**

**Thin lens equation**

*Position of the waists before and after a lens*

**- Beam waist ratio**

\[
m = \frac{w_0'}{w_0} = \left[1 + \frac{s}{f'} \right]^2 + \left(\frac{z_R}{f'}\right)^2 \right]^{-1/2}
\]

\[
z_{R'} = m^2 z_R
\]

1. In the case \( s \) tends to infinity, \( m = 0 \rightarrow w' = 0 \) (but \textbf{NO!} theoretical the limit of \( w \sim \lambda/2 \))

2. In the case \( s = -f' \) (the waist is at the objet focal plan), so \( m = f/z_R = f\lambda/\pi w_0^2 \)

The situation is very different from the GO!

\[
\theta' = \frac{\lambda}{\pi w_0'} = \frac{w_0}{f'}
\]

Example: \( w_0 = 1 \text{ cm}, f = 0.1 \text{ m}, \lambda = 0.6328 \text{ µm} \rightarrow z_R = 500 \text{ m}, \theta = 0.001^\circ \)

\( s = -f' = -0.1 \text{ m} \rightarrow s' = 0.1 \text{ m}, w'_0 = 2 \text{ µm}, \theta' = 5.7^\circ \)

We pass from a large waist to a small one ...

In practice, \( w_0 \) can be very small, so that \( w'_0 \) is very large.

We find of course all the intermediate cases ...