Lecture at Xidian University
on Frontiers in modern optics

Scattering of shaped beam by particles and its applications

I. Fundamentals of light scattering by small particles

Kuan Fang REN

CORIA/UMR 6614 CNRS - Université et INSA de Rouen
School of physics and optoelectronic Eng., Xidian University

西安电子科技大学
现代光学前沿专题

波束散射理论和应用

第一讲：小粒子光散射基础

任宽芳

法国鲁昂大学 — CORIA研究所
西安电子科技大学物理与光电学院

个人网页：http://ren.perso.neuf.fr
电话：中国 8820 2673， 法国 02 32 95 37 43

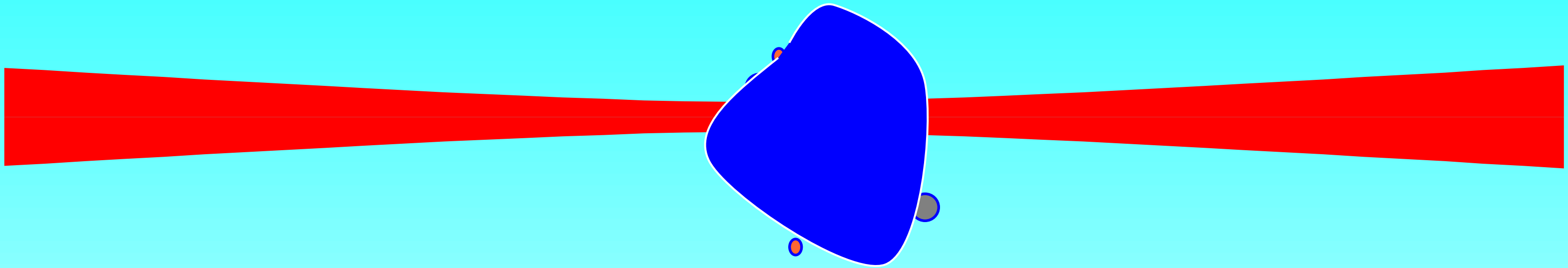
Plan of lecture

- Introduction
- Fundamentals
- Maxwell equations and wave equations
  - Scalar and vector wave functions
- Solutions of wave equations

UNIVERSITÉ DE ROUEN

CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Introduction



- **Elastic scattering**
- **Inelastic scattering**

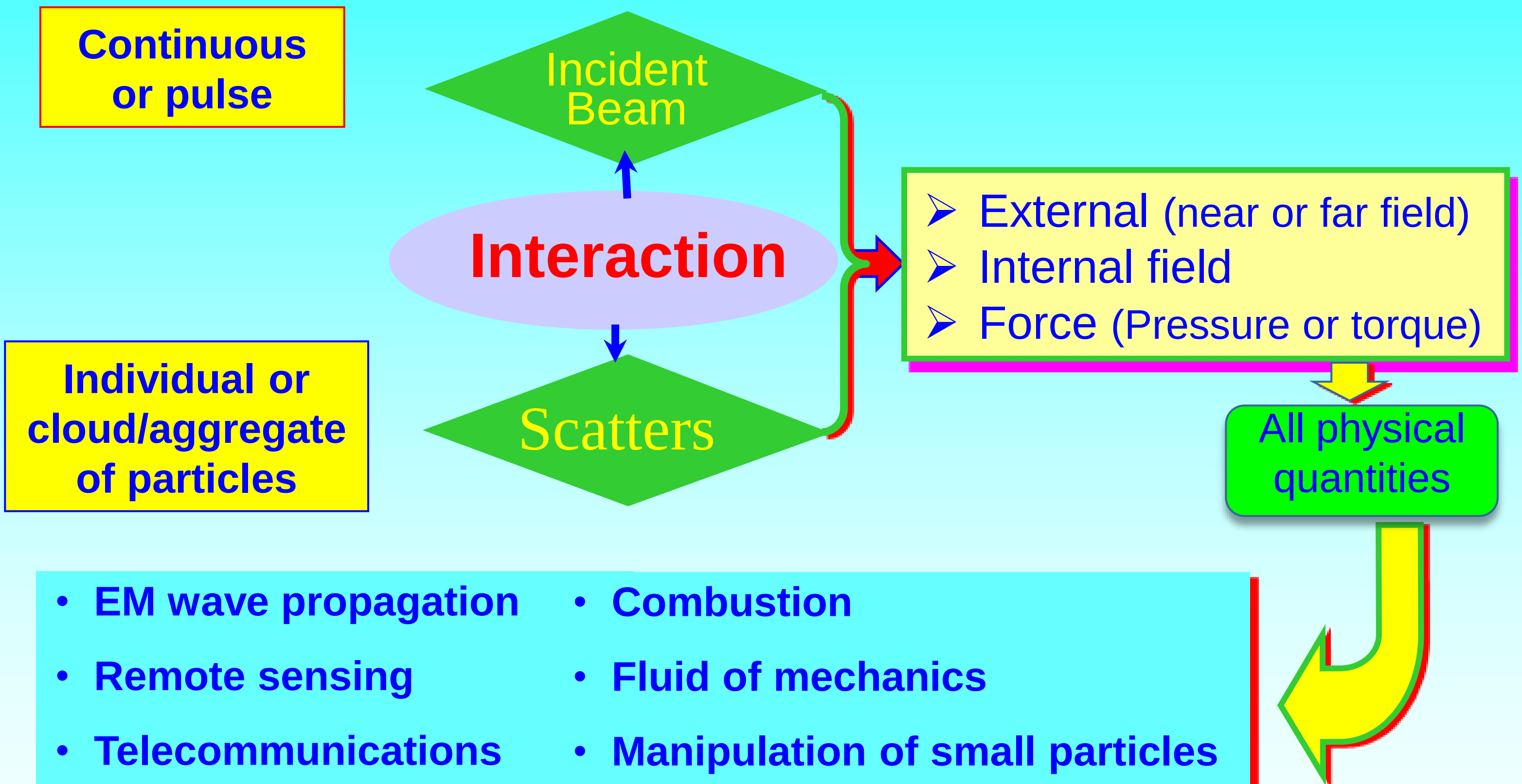
- **Single scattering**
- **Multiple scattering**
- **Coherent scattering**

The effects of multiple scattering and coherent scattering depend not only on the concentration but also the size of the particles.

UNIVERSITÉ DE ROUEN

coria UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Introduction



- **EM wave propagation**
- **Remote sensing**
- **Telecommunications**
- **Environmental science**
- **Combustion**
- **Fluid of mechanics**
- **Manipulation of small particles**
- **Biological and life science**

Introduction

Theoretical models

- **Rigorous theories**
  - **Lorenz-Mie Theory**
  - **Generalized Lorenz-Mie theory(GLMT)**
  - . . .
- **Numerical methods (mainly for non-spherical object)**
  - FDTD - Finite Difference Time Domain
  - MoM - Method of Moments
  - FEM – Finite Element Method
  - T-Matrix
  - DDA - Dipole Discrete Approximation (ADDA and DDSCAT)
  - . . .

Introduction

Theoretical models

Approximate models

- Rayleigh's theory： any shape, dimension $l << \lambda$
- Rayleigh-Gans： $|m - 1| <<1$
- Diffraction： $l \sim \lambda$
- Geometrical Optics: $l >> \lambda$
- Geometrical Theory of Diffraction
- **Ray theory of wave (RTW)** **under development**
- . . .

Introduction

Applications

Physics

Understanding of physical procedures

Astrophysics

Effect of force

Interplanetary Dust Particles

EM Scattering of an arbitrary object

Optical metrology

Environment

Energy

… …

EM wave

propagation & detection

communication

Radar …

Optical Tweezers

Biological/life science

Micro-fluidics

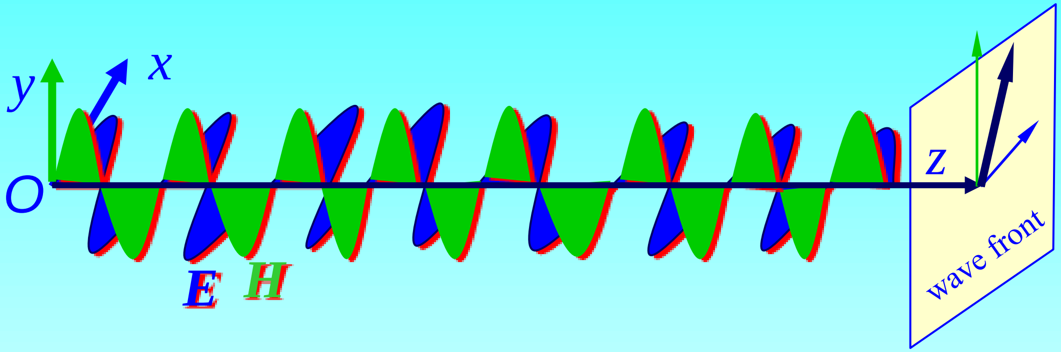
Material science

UNIVERSITÉ DE ROUEN

coRia UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Fundamentals - Plane wave

Electromagnetic (EM) field and properties



Wave front= plane || xOy
Propagation direction ⊥ E et H

Wave vector: $\boldsymbol{k}$

Wave number: $k = \dfrac{2\pi}{\lambda}$

Two polarizations: $\begin{cases} E_x = A_x \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi_0) \\ E_y = A_y \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi_0) \end{cases}$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \vec{k}.\vec{r} - \phi_0)}$$ (complex fonction)

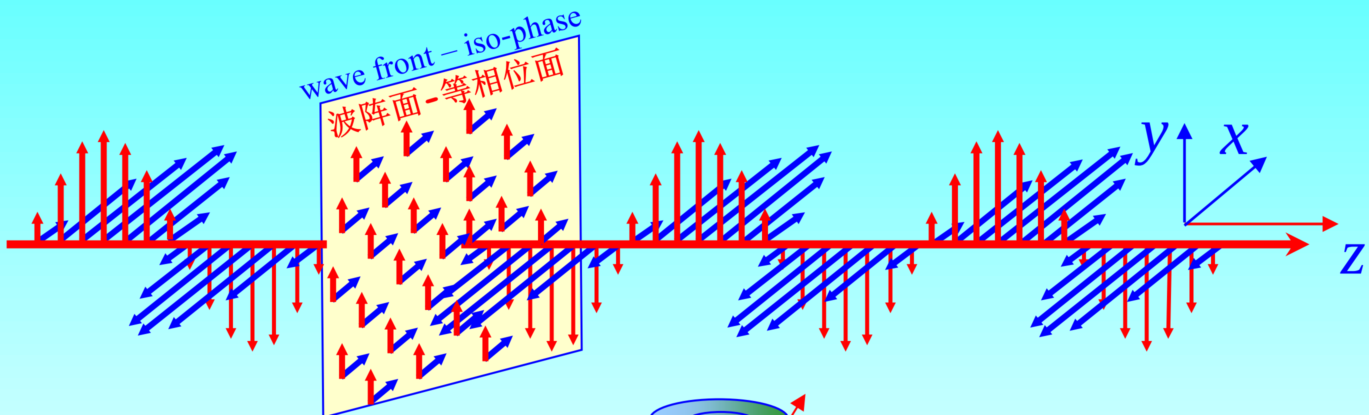
In an isotropic medium:

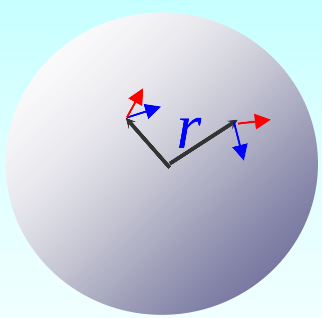
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H} \qquad \boldsymbol{H} = \frac{1}{\mu\omega}\boldsymbol{k} \wedge \boldsymbol{E}$$

$\boldsymbol{E}$ –electric field
$\boldsymbol{H}$ –magnetic field
ε – permittivity
μ - permeability

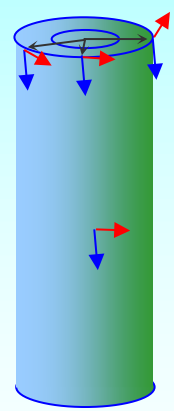
Fundamentals - Different forms of wave

Electromagnetic (EM) field and properties

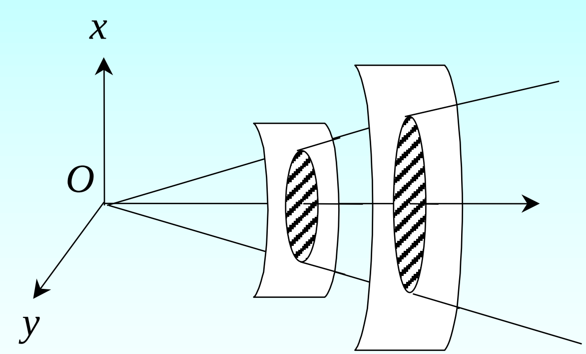




spherical



Cylindrical



Plane wave in far field

Fundamentals - Refractive index

Complex refractive index

$$\tilde{m} = m_r - m_i i$$

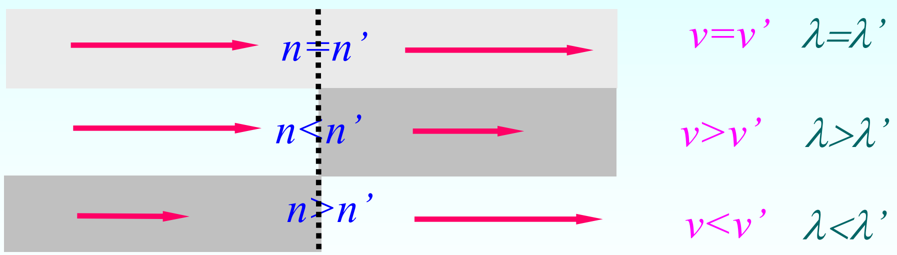
Real part - velocity: $$m_r = \frac{c}{v}$$

Examples:

vacuum: $c = 3\times10^8$ ms^{-1} , λ=0,6328 µm

water： n_{eau}=1,33 v_{eau}=2,26×10^8 ms^{-1} , λ_{eau}=0,4758 µm

glass： n_{verre}=1,5 v_{verre}=2,00×10^8 ms^{-1} , λ_{verre}=0,4219 µm



Fundamentals - Refractive index

Complex refractive index

Imaginary part - absorption:

穿透深度: $$d = \frac{1}{m_i k_0} = 0.16\frac{\lambda}{m_i}$$

$$E = E_0 e^{i(\omega t - n k_0 z + \phi)}$$
$$= E_0 e^{i\omega t - i m_r k_0 z - m_i k_0 z + i\phi}$$
$$= E_0 e^{-m_i k_0 z} e^{i(\omega t - m_r k_0 z + \phi)}$$

Amplitude à z:

$$E_0(z) = E_0(z=0)e^{-m_i k_0 z}$$

Penetration depth d:

$$\frac{E_0(z=d)}{E_0(z=0)} = e^{-1}$$

i.e. $\frac{I(d)}{I(0)} = \frac{1}{e^2} = 13.5\%$

$$\Rightarrow d = \frac{1}{m_i k_0} = 0.16\frac{\lambda}{m_i}$$

λ = 0.6328 µm
m_i = 0.1, d = 1 µm
m_i = 0.0001, d = 1 mm

Fundamentals - Energy and momentum

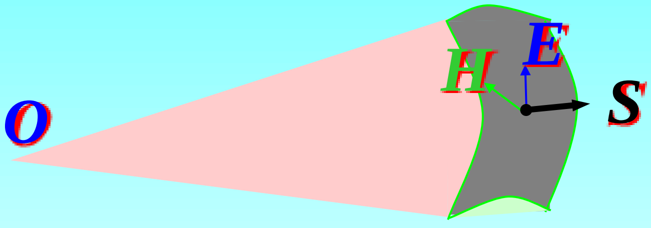
Poynting's vector and Intensity

Energy density (J/m^3): $u = \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}\right)$

Poynting's vector (W/m^2):

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right)$$

Complex function



In isotropic medium

Poynting's vector: $\boldsymbol{S} = vu\boldsymbol{n}$

Intensity: $I = \left\|\boldsymbol{S}\right\| \propto E^2$

Fundamentals - Energy and momentum

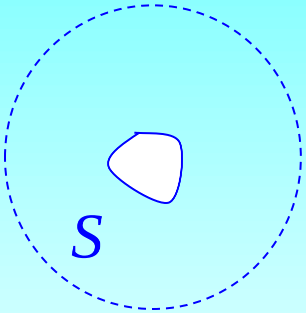
Stress tensor, force and torque

Stress tensor:

$$\overleftrightarrow{T} = \frac{1}{2}\mathrm{Re}\left[\varepsilon \boldsymbol{E}\boldsymbol{E}^* + \mu \boldsymbol{H}\boldsymbol{H}^* + \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{E}^* + \boldsymbol{H}\cdot\boldsymbol{H}^*\right)\overleftrightarrow{I}\right]$$

Radiation force: $$\boldsymbol{F} = \oint_S d\boldsymbol{S}\langle\overleftrightarrow{T}\rangle$$

Torque: $$\boldsymbol{M} = -\oint_S d\boldsymbol{S}\cdot\left(\langle\overleftrightarrow{T}\rangle \times \boldsymbol{r}\right)$$



Integration over a sphere including the particle:

- when $r\to\infty$, $E_r\to 0$:

$$\boldsymbol{F} = -\frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\mathrm{Re}\left[\epsilon(|E_\theta|^2+|E_\phi|^2)+\mu(|H_\theta|^2+|H_\phi|^2)\right]\boldsymbol{e}_r r^2\sin\theta d\theta d\phi$$

- but E_r can never be neglected for torque:

$$\boldsymbol{M} = -\frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\mathrm{Re}\left[\left(\epsilon E_r E_\phi^* + \mu H_r H_\phi^*\right)\boldsymbol{e}_\theta - \left(\epsilon E_r E_\theta^* + \mu H_r H_\theta^*\right)\boldsymbol{e}_\phi\right] r^3\sin\theta d\theta d\phi$$

UNIVERSITÉ DE ROUEN

CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Fundamentals - Scattering matrix

Relation between incident and scattered waves
in far field

$$\begin{pmatrix} E_{\parallel s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{\parallel i} \\ E_{\perp i} \end{pmatrix}$$

For a particle of spherical symmetry:
$S_3 = S_4 = 0$

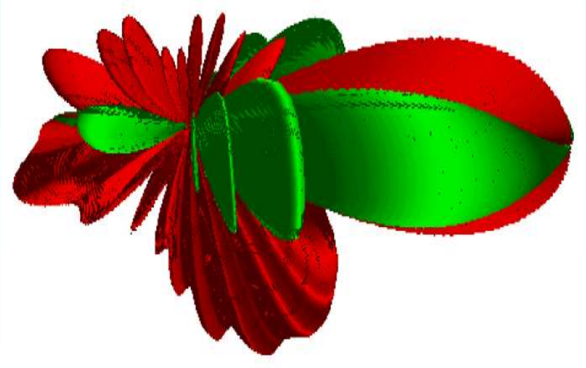
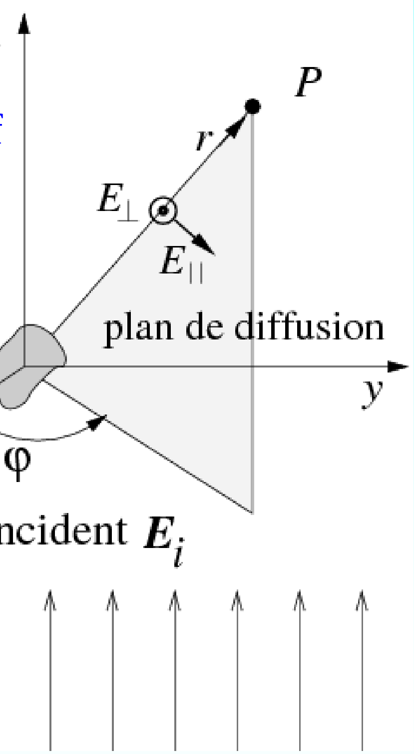


Diagramme de diffusion d'une sphère éclairée par un faisceau gaussien (hors axe), polarisation perpendiculaire en rouge et parallèle en vert.

Definition of polarization

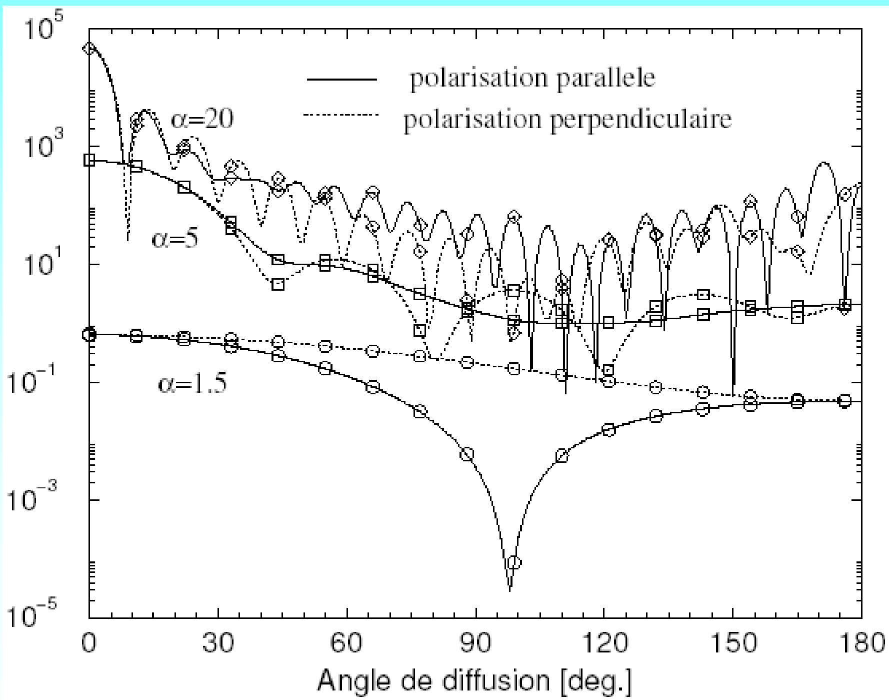


Fundamentals - Phase function

Scattering diagram

sphere: *d*=2*a*， refraction index: *m*=1.33

Incident plane wave



$$I(\theta,\varphi)=\frac{I_0F(\theta,\varphi)}{k^2r^2}$$

Incident wave polarized in *x* direction:

$$I \;=F(\theta,\phi=0)=\left|S_2\right|^2$$

$$I \;=F(\theta,\phi=90°)=\left|S_1\right|^2$$

Particle size parameter:

$$\alpha=\frac{\pi d}{\lambda}$$

Fundamentals - Cross sections

Integral properties of a scatter

External EM field: $\boldsymbol{E}=\boldsymbol{E}_i+\boldsymbol{E}_s$, $\boldsymbol{H}=\boldsymbol{H}_i+\boldsymbol{H}_s$

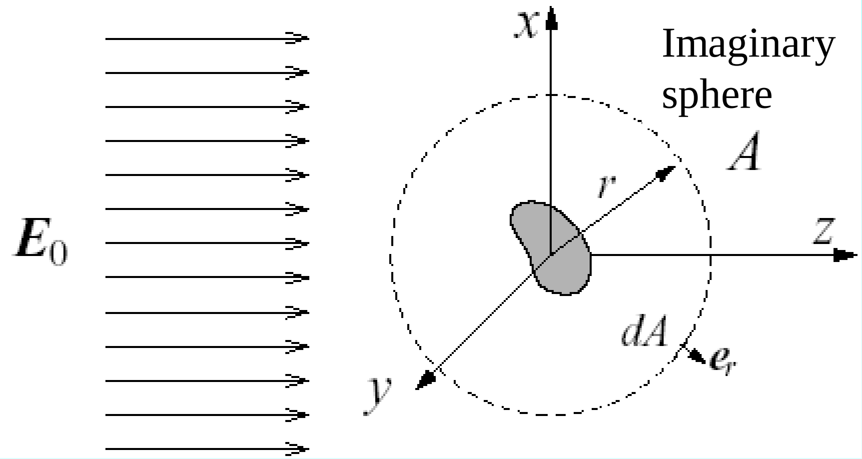
Poynting vector of total filed:

$$\boldsymbol{S}=\frac{1}{2}Re\{\boldsymbol{E}\times\boldsymbol{H}^*\}=\boldsymbol{S}_i+\boldsymbol{S}_s+\boldsymbol{S}_{ext}$$

$$\boldsymbol{S}_i=\frac{1}{2}Re\{\boldsymbol{E}_i\times\boldsymbol{H}_i^*\}$$

$$\boldsymbol{S}_s=\frac{1}{2}Re\{\boldsymbol{E}_s\times\boldsymbol{H}_s^*\}$$

$$\boldsymbol{S}_{ext}=\frac{1}{2}Re\{\boldsymbol{E}_i\times\boldsymbol{H}_s^*+\boldsymbol{E}_s\times\boldsymbol{H}_i^*\}$$



Energy balance

$$W_{abs}=-\int_A \boldsymbol{S}\cdot\boldsymbol{e}_r dA=W_{inc}-W_{sca}+W_{ext}$$

$$W_{inc}=-\int_S \boldsymbol{S}_i\cdot\boldsymbol{e}_r dA,\ \ W_{sca}=-\int_S \boldsymbol{S}_s\cdot\boldsymbol{e}_r dA,\ \ W_{ext}=-\int_S \boldsymbol{S}_{ext}\cdot\boldsymbol{e}_r dA$$

$$\boldsymbol{W}_{ext}=\boldsymbol{W}_{abs}+\boldsymbol{W}_{sca}$$

UNIVERSITÉ DE ROUEN

coria UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Fundamentals - Cross sections

Definition of cross sections and efficient factors

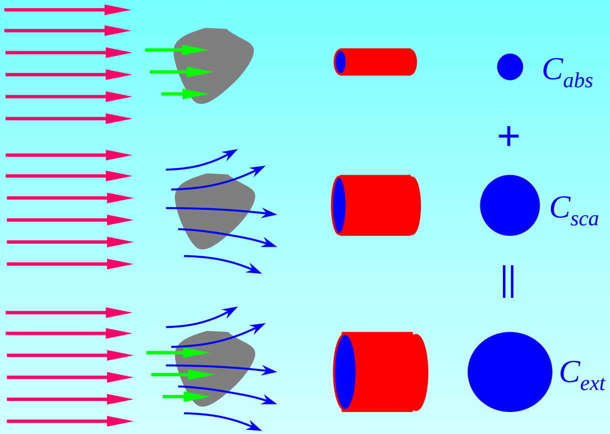
Efficient Sections:

Absorption section: $C_{abs} = \dfrac{W_{abs}}{I_i}$

Scattering section: $C_{sca} = \dfrac{W_{sca}}{I_i}$

Extinction sction: $C_{ext} = \dfrac{W_{ext}}{I_i}$

Physical interpretation



Absorption
+
Scattering
||
Total perturbation

Efficiency factors

$$Q_{ext} = \frac{C_{ext}}{A}, \quad Q_{abs} = \frac{C_{abs}}{A}, \quad Q_{sca} = \frac{C_{sca}}{A}$$

Transparent particle

$$C_{abs} = 0, \quad C_{ext} = C_{sca}$$

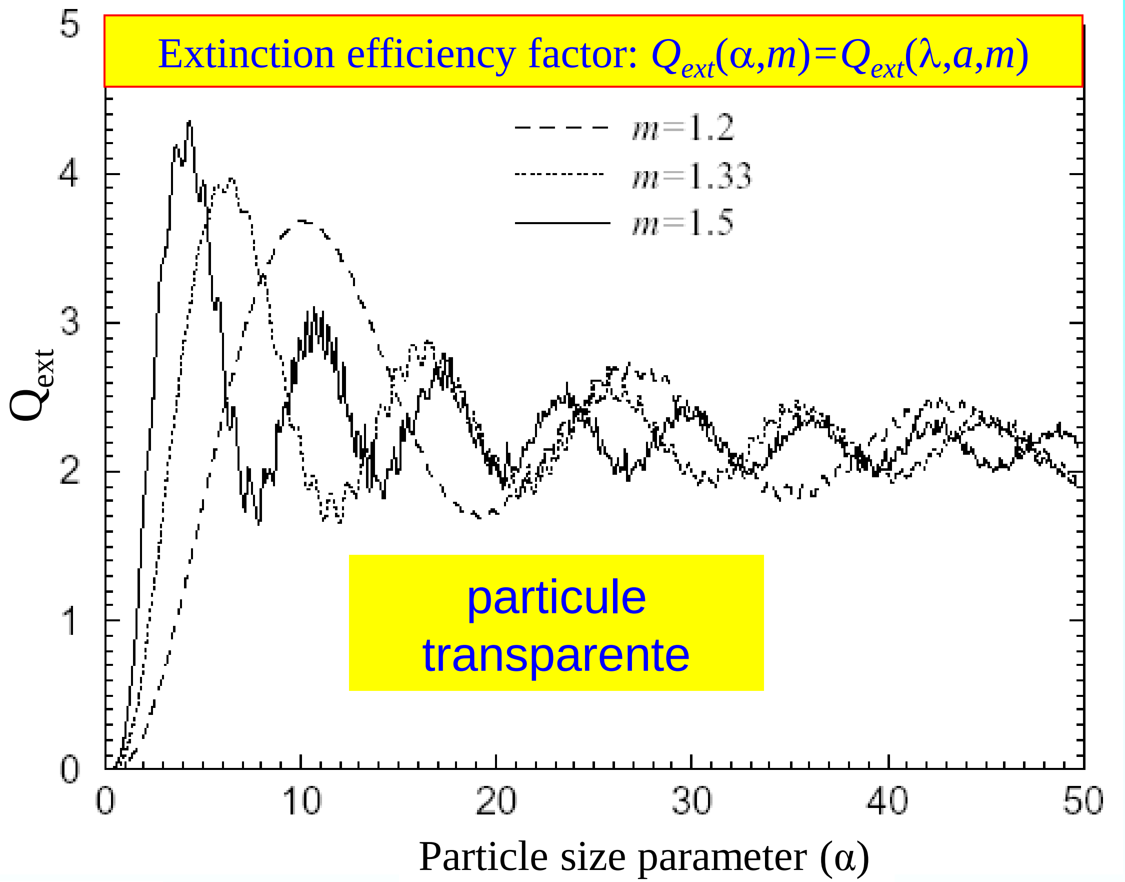
$$Q_{abs} = 0, \quad Q_{ext} = Q_{sca}$$

UNIVERSITÉ DE ROUEN

coRia UMR 6614 CNRS - UNIVERSITÉ et INSA de ROUEN

Fundamentals - Cross sections

Know to read and use the graph



Small particle

$$Q_{ext} \sim \frac{d^4}{\lambda^4}$$

Large particle

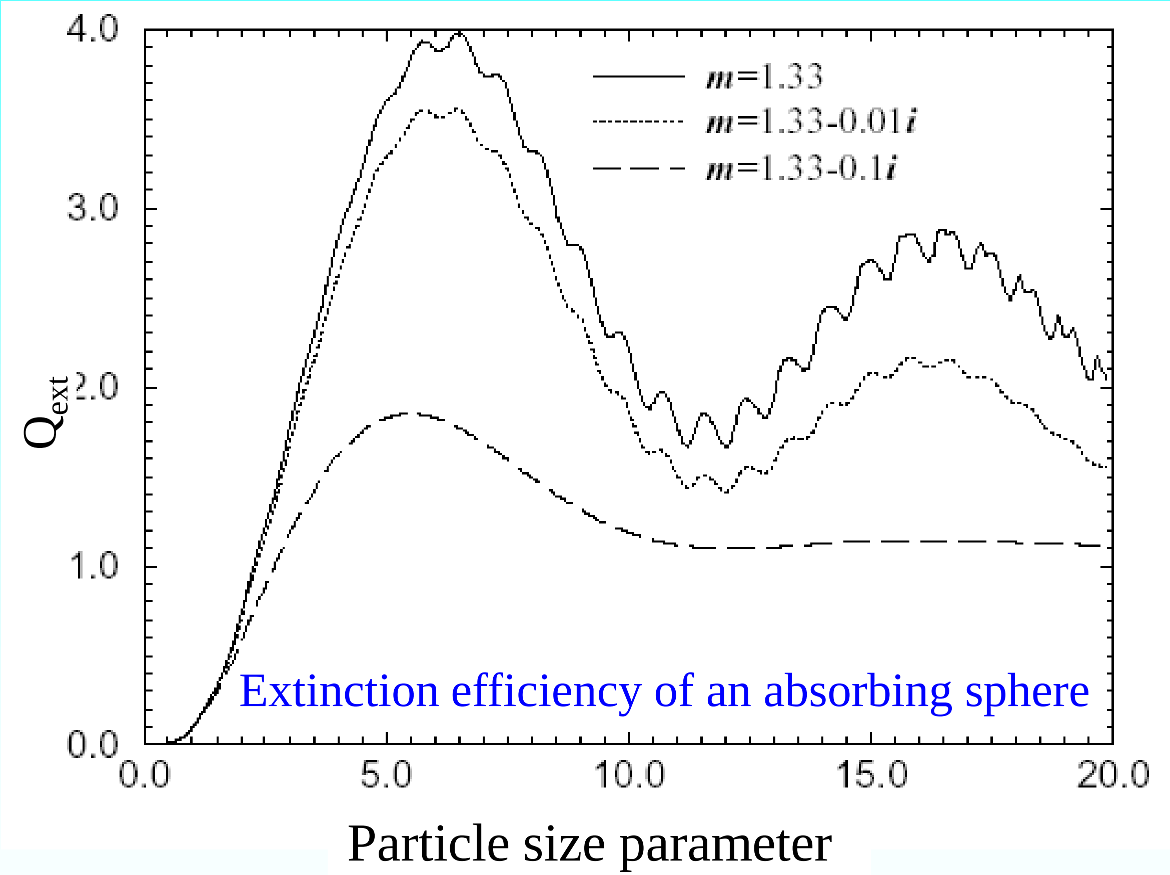
$$Q_{ext} \to 2$$

The high frequency is smoothened for polydisperse distribution

Why the sky is blue and the sun is red at rising and sunset.

Fundamentals - Cross sections

Know to read and use the graph



Extinction efficiency of an absorbing sphere

Small particle

$d \ll \lambda$:

$$Q_{ext} = \frac{8}{3}\left(\frac{\pi d}{\lambda}\right)^4 \mathrm{Re}\left(\frac{m^2-1}{m^2+2}\right)$$

UNIVERSITÉ DE ROUEN

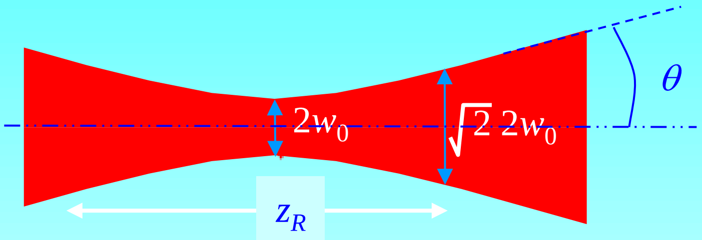
CORIA UMR 6614 CNRS - UNIVERSITÉ et INSA de ROUEN

Fundamentals – Gaussian beam

Characteristics of a beam

(a). Intensity： decreasing along *z* and *r*.

$$I(r,z)=I_0\left[\frac{w_0}{w(z)}\right]^2\exp\left[-\frac{2r^2}{w^2(z)}\right]$$



(b). Beam waist radius $w_0=w(z=0)$:

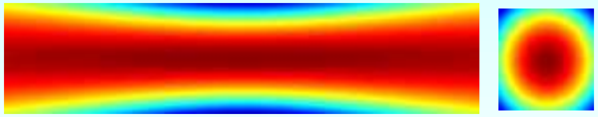
$$I(r=w)=\frac{I(r=0)}{e^2}$$

$$w(z)=w_0\sqrt{1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2}$$

$$z_R=\frac{\pi w_0^2}{\lambda}$$

(c). Divergence angle

$$\theta=\lim_{z\to\infty}\left[\arctan\left(\frac{w(z)}{z}\right)\right]=\arctan\left(\frac{\lambda}{\pi w_0}\right)$$



(d). Rayleigh distance: $z_R = \pi w_0^2/\lambda$

$$I(0,z_R)=\frac{I_0}{2},\qquad w(z_R)=\sqrt{2}w_0$$

Fundamentals – Gaussian beam

Examples:

Divergence:

1. λ= 600 nm :

w_0 = 10 µm: z_R = 500 µm, θ = 0.02 rad

w_0 = 1 mm: z_R = 5 m, θ = 0.0002 rad

w_0 = 1 cm: z_R = 500 m, θ = 0.00002 rad

2. w_0= 100 µm :

λ = 10.6 µm (CO2): z_R = 3 mm, θ = 0.034 rad

λ = 0.6328 µm (He-Ne): z_R = 5 cm, θ = 0.002 rad

λ= 0.488 µm (bleu YAG): z_R = 6.4 cm, θ = 1.5 mrad

Intensity:

Intensity at the center:

$$I=I_0$$

Intensity at the border:

$$I(r=w)=\frac{I(r=0)}{e^2}=\frac{I_0}{e^2}$$

Total power of the beam:

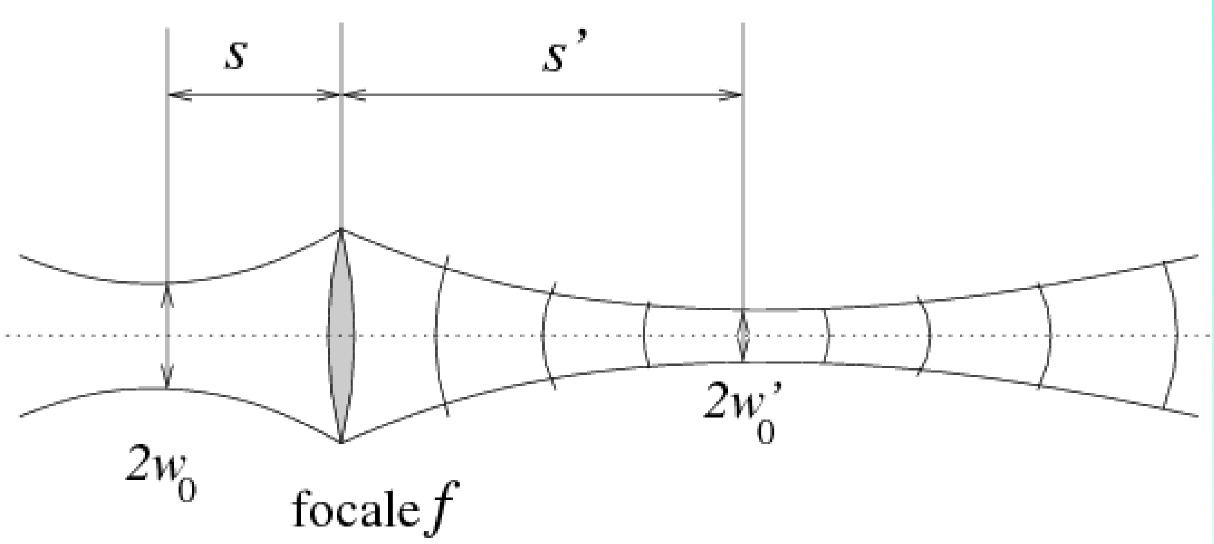
$$P=\int I(r,z)dS=\int_0^{2\pi}\int_0^{\infty} I(r,z)rdrd\theta$$

$$=2\pi I_0\left[\frac{w_0}{w(z)}\right]^2\int_0^{\infty}\exp\left[-\frac{2r^2}{w^2(z)}\right]rdr$$

$$=I_0\frac{\pi w_0^2}{2}$$

Fundamentals – Gaussian beam

Conjugation relation of a Gaussian beam



$$\frac{1}{s'}-\frac{1}{s+\frac{z_R^2}{s+f'}}=\frac{1}{f'}$$

or

$$s'=f'\left[1-\frac{f'(s+f')}{(s+f')^2+z_R^2}\right]$$

Fundamentals – Gaussian beam

Conjugation relation of a Gaussian beam

Special cases:

$$\frac{1}{s'}-\frac{1}{s+\dfrac{z_R^2}{s+f'}}=\frac{1}{f'}$$

• Collimated beam: $z_R, \rightarrow \infty$ so that $s' = f'$.

Same result as a thin lens: parallel wave focused to the focal.

• Incident beam waist at focal, exit beam waist also,
since $s = -f$ => $s'=f'$. Result completely different from that of GO.

• The positions of the exit beam waist depend not only on *s* and *f* but also on z_R (so λ and w_0).

Fundamentals – Gaussian beam

Magnification of beam waist

$$m = \frac{w_0'}{w_0} = \left[\left(1 + \frac{s}{f'}\right)^2 + \left(\frac{z_R}{f'}\right)^2 \right]^{-1/2} \qquad z_R' = m^2 z_R$$

1. Incident plane wave ($z_R = \infty$), $m = 0$ → w’=0

   Attention: this is impossible, **theoretical limit:** $w_0 \sim \lambda/2$

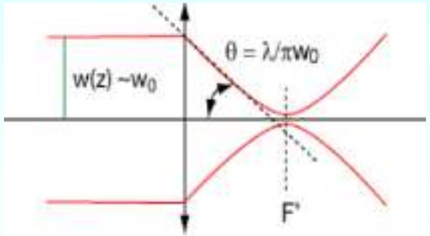
2. When $s = -f'$ (incident beam waist at focal) , $m = f/z_R = f\lambda / \pi w_0^2$

   Divergence angle:

$$\theta' = \lambda / \pi w_0' = w_0/f'.$$

Examples: w_0 = 1 cm, f' = 0.1 m, λ=0.6328 μm → z_R = 500 m, θ = 0.001°

s=–f’=–0.1 m → s’=0.1 m, w'_0= 20 μm, θ'= 5.7°



Large divergence becomes small one and vice-versa。

Applications: expansion of a beam.

Maxwell equations & wave equations

1. Maxwell equations in differential form

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{H} &= \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \\ \nabla \cdot \boldsymbol{D} &= \rho \\ \nabla \cdot \boldsymbol{B} &= 0 \end{aligned}$$

E: electric field
H: magnetic field
D: electric displacement
B: magnetic induction
j: electric current density
ρ: electric charge density

2. Constitutive relations

$$\begin{aligned} \boldsymbol{D} &= \epsilon \boldsymbol{E} \\ \boldsymbol{B} &= \mu \boldsymbol{H} \end{aligned}$$

ε : permittivity
μ: permeability

ε and μ are scalars in an isotropic medium, matrix in an anisotropic medium.

Maxwell equations & wave equations

3. Wave equations of a harmonic wave in free space:

- Harmonic wave: $\boldsymbol{A}(\boldsymbol{r},t)=\boldsymbol{A}(\boldsymbol{r})e^{i\omega t}$ ($\boldsymbol{A}$ stands for $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{H}$ or $\boldsymbol{B}$)
- Free space: $\boldsymbol{j}=0$, $\rho=0$

By calculating the curl of the first two Maxwell equations, using the 3rd and 4th equations and the identity:

$$\nabla\times(\nabla\times\boldsymbol{A})=\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$$

We obtain the wave equations

$$\begin{aligned}\nabla^2\boldsymbol{E}+k^2\boldsymbol{E}&=0\\ \nabla^2\boldsymbol{H}+k^2\boldsymbol{H}&=0\end{aligned}$$

To be checked by yourself.

with $k^2=\omega^2\mu\varepsilon$

- k is the wave number
- ω is the angular frequency
- $v=\sqrt{\mu\varepsilon}$ is the wave velocity

UNIVERSITÉ DE ROUEN — CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Maxwell equations & wave equations

4. Scalar wave equations

- Plane wave: It is evident that the plane wave $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0 e^{i(\omega t \pm \boldsymbol{k}\cdot\boldsymbol{r})}$ is a solution of the wave equation.
- General cases: we can show that in free space **two independent scalar functions are sufficient to describe all EM waves**. The two scalar functions can be two components of a vector potential.
- Hertz vectors: We choose often a component of the electric Hertz vector Π_e and a component of the magnetic Hertz vector Π_m as the independent scalar functions and construct the two Hertz vectors:

$$\boldsymbol{\Pi}_e = \boldsymbol{a}\Pi_e \quad \text{and} \quad \boldsymbol{\Pi}_m = \boldsymbol{a}\Pi_m$$

They satisfy the same wave equation:

$$(\nabla^2 + k^2)\Pi = 0$$

and the EM fields are given by:

$$\begin{aligned} \boldsymbol{E} &= \nabla\times(\nabla\times\boldsymbol{\Pi}_e) - i\omega\mu\nabla\times\boldsymbol{\Pi}_m \\ \boldsymbol{H} &= i\omega\varepsilon\nabla\times\boldsymbol{\Pi}_e + \nabla\times(\nabla\times\boldsymbol{\Pi}_m) \end{aligned}$$

Maxwell equations & wave equations

5. Vector wave equations

- Vector wave equation (***A*** stands for ***E*** or ***H*** or Π)

$$(\nabla^2 + k^2)\boldsymbol{A} = 0$$

- Vector wave functions (Π stands for either Π_e or Π_m):

We suppose that the scalar function Π satisfies the scalar Helmholtz equation and ***a*** is a vector constant. Then we compose

$$\begin{aligned} \boldsymbol{L} &= \nabla\Pi \\ \boldsymbol{M} &= \nabla\times(\boldsymbol{a}\Pi) \\ \boldsymbol{N} &= \frac{1}{k}\nabla\times\boldsymbol{M} \end{aligned}$$

with properties

$$\begin{aligned} \nabla\times\boldsymbol{L} &= 0 \\ \nabla\cdot\boldsymbol{M} &= 0 \\ \nabla\cdot\boldsymbol{N} &= 0 \end{aligned}$$

$$\nabla\cdot\boldsymbol{L} = \nabla^2\Pi = -k^2\Pi$$

- EM fields:

$$\boldsymbol{E} = \sum_n (A_n\boldsymbol{N}_n + B_n\boldsymbol{M}_n)$$

$$\boldsymbol{H} = \frac{k}{i\omega\mu}\sum_n (A_n\boldsymbol{M}_n + B_n\boldsymbol{N}_n)$$

Cf. Bohren p198

Check this writing from ***E***.

The divergences of ***E*** and ***H*** are null in free space, so no ***L***.

Solution of wave equations

1. General description

Our task is now to solve the scalar wave equation

$$(\nabla^2 + k^2)\Pi = 0$$

in different coordinate systems by the variable separation method.

i. Wave functions : $\Pi(x_1, x_2, x_3) = X_1(x_1)X_2(x_2)X_3(x_3)$

ii. Differential wave equations

$$D(\Pi(x_1, x_2, x_3)) = 0 \quad \Rightarrow \quad \begin{aligned} D(X_1(x_1)) &= \mu \\ D(X_2(x_2)) &= \nu \\ D(X_3(x_3)) &= f(\mu, \nu) \end{aligned}$$

iii. Solutions:

$$\Pi(x_1, x_2, x_3) = \sum_{\mu\nu} C_{\mu\nu} X_1(x_1)X_2(x_2)X_3(x_3)$$

$C_{\mu\nu}$:
Beam shape coeff.
Scattering coeff.
Internal field coeff.

$X_1(x_1)X_2(x_2)X_3(x_3)$:
Usually one or two special functions

UNIVERSITÉ DE ROUEN
coria UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Solution of wave equations

2. Solution in the cylindrical coordinate system

- Differential equations in the spherical coordinate system

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Pi}{\partial\phi^2}+\frac{\partial^2\Pi}{\partial z^2}+k^2\Pi=0$$

- Separation of the variables. We suppose:

$$\Pi(\rho,\phi,z)=R(\rho)\Phi(\phi)Z(z)$$

and obtain:

$$\frac{d^2Z}{dz^2}+h^2Z=0$$

$$\frac{d^2\Phi}{d\phi^2}+\nu^2\Phi=0$$

Harmonic oscillator differential equations.

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\left(\mu^2-\frac{\nu^2}{\rho^2}\right)R=0$$

Bessel differential eq.

with $h^2+\mu^2=k^2$

Solution of wave equations

- General solutions of these three differential equations are respectively
  - Exponential function: $\Phi_m(\phi) = e^{im\phi}$ *m* is the azimuth mode.
  - Cylindrical Bessel function: $R_m(\mu\rho) = Z_m(\mu\rho)$
  - Exponential function: $Z_h(z) = e^{-ihz}$ *h* is k_z.

So the general solution is given by

$$\Pi_{mh} = Z_m(\mu\rho)e^{i(m\phi - hz)}$$

The Hertz potential for plane wave (*h* const.) is given by

$$\Pi = \sum_{m=-\infty}^{\infty} c_m Z_m(\mu\rho)e^{i(m\phi - hz)}$$

For shaped beam:

$$\Pi = \sum_{m=-\infty}^{\infty} \int_h c_{mh} Z_m(\mu\rho)e^{i(m\phi - hz)}\, dh$$

Different fields are expressed with different coefficients and adequate Bessel function.

UNIVERSITÉ DE ROUEN

CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Solution of wave equations

- Electromagnetic field

With help of the relation between the Hertz vectors and EM fields:

$$\boldsymbol{E} = \nabla\times(\nabla\times\boldsymbol{\Pi}_e) - i\omega\mu\nabla\times\boldsymbol{\Pi}_m$$

$$\boldsymbol{H} = i\omega\varepsilon\nabla\times\boldsymbol{\Pi}_m + \nabla\times(\nabla\times\boldsymbol{\Pi}_m)$$

We choose $\boldsymbol{\Pi}=\Pi\boldsymbol{e}_z$ and establish

$$\begin{aligned}
E_\rho &= \frac{\partial^2\Pi_e}{\partial\rho\partial z} - \frac{i\omega\mu}{\rho}\frac{\partial\Pi_m}{\partial\phi}\\
E_\phi &= \frac{1}{\rho}\frac{\partial^2\Pi_e}{\partial\phi\partial z} + i\omega\mu\frac{\partial\Pi_m}{\partial\rho}\\
E_z &= \frac{\partial^2\Pi_e}{\partial z^2} + k^2\Pi_e\\
H_\rho &= \frac{\partial^2\Pi_m}{\partial\rho\partial z} + \frac{i\omega\epsilon}{\rho}\frac{\partial\Pi_e}{\partial\phi}\\
E_\phi &= \frac{1}{\rho}\frac{\partial^2\Pi_m}{\partial\phi\partial z} - i\omega\epsilon\frac{\partial\Pi_e}{\partial\rho}\\
E_z &= \frac{\partial^2\Pi_m}{\partial z^2} + k^2\Pi_m
\end{aligned}$$

by using:

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\boldsymbol{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\boldsymbol{e}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\boldsymbol{e}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right]\boldsymbol{e}_z$$

To be checked by yourself.

UNIVERSITÉ DE ROUEN

CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN

Solution of wave equations

- Vector wave functions:

By introducing the Hertz function in the above equations, the EM fields can be expressed as vector wave function in the form like:

$$\boldsymbol{m} = \nabla \times (\Pi_z \boldsymbol{e}_z)$$

$$\boldsymbol{E} = \sum_{m=-\infty}^{\infty} (A_m \boldsymbol{m}_m + B_m \boldsymbol{n}_m)$$

with

$$\boldsymbol{m}_{mh} = \left[\frac{im}{\rho} Z_m(\mu\rho)\boldsymbol{e}_\rho - \frac{\partial Z_m(\mu\rho)}{\partial\rho}\boldsymbol{e}_\phi\right] e^{i(m\phi - hz)}$$

$$\boldsymbol{n}_{mh} = \frac{1}{k}\left[-ih\frac{\partial Z_m(\mu\rho)}{\partial\rho}\boldsymbol{e}_\rho - \frac{hm}{\rho} Z_m(\mu\rho)\boldsymbol{e}_\phi + \mu^2 Z_m(\mu\rho)\boldsymbol{e}_z\right] e^{i(m\phi - hz)}$$

- In classical Lorentz Mie theory – scattering of a plane wave by a cylindrical particle, for a given incident angle ζ, $h=\cos\zeta$ is constant. Only the summation on m is necessary. *But in beam scattering the incident wave and scattered wave* ***must be expanded*** *in h, so a integral on h is necessary.*

Solution of wave equations

3. Solution in the spherical coordinate system

- Differential equations in the spherical coordinate system

For convenience we note $\Pi = r\varphi$, then the wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}+k^2\varphi=0$$

- Separation of the variables. We suppose:

$$\varphi(r,\theta,\phi)=R(kr)\Theta(\theta)\Phi(\phi)$$

and obtain: $x=\cos\theta,\quad \nu=n(n+1)$

$$\frac{d\Phi}{d\phi}+m^2\Phi=0$$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left(\nu^2-\frac{m^2}{1-x^2}\right)P=0$$

$$\frac{d^2R}{d(kr)^2}-\frac{2}{kr}\frac{dR}{d(kr)}+\left[1-\frac{n(n+1)}{(kr)^2}\right]R=0$$

Solution of wave equations

- General solutions of these three differential equations

$$\frac{d\Phi}{d\phi} + m^2\Phi = 0 \qquad (1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\nu^2 - \frac{m^2}{1-x^2}\right)P = 0 \qquad \frac{d^2R}{d(kr)^2} - \frac{2}{kr}\frac{dR}{d(kr)} + \left[1 - \frac{n(n+1)}{(kr)^2}\right]R = 0$$

Are respectively

- Exponential function: $e^{im\phi}$
- (Associated) Legendre function: $P_n^m(\cos\theta)$
- Spherical Bessel function: $z_n(kr)$

So the general solution is given by

$$\varphi_{nm} = z_n(kr)P_n^m(\cos\theta)e^{im\phi}$$

The Hertz potential for any EM wave is given by

$$\Pi(r,\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n} c_{nm}\begin{pmatrix}\psi_n(kr)\\ \chi_n(kr)\\ \xi_n(kr)\end{pmatrix} P_n^{|m|}(\cos\theta)e^{im\phi}$$

$$\psi_n(x) = xj_n(x)$$
$$\chi_n(x) = xh_n^{(1)}(x)$$
$$\xi_n(x) = xh_n^{(2)}(x)$$

Different fields are expressed with different coefficients and adequate Bessel function.

UNIVERSITÉ DE ROUEN — CORIA UMR 6614 CNRS - UNIVERSITÉ et INSA de ROUEN

Solution of wave equations

- Electromagnetic field

With help of the relation between the Hertz vectors and EM fields:

$$\boldsymbol{E} = \nabla\times(\nabla\times\boldsymbol{\Pi}_e) - i\omega\mu\nabla\times\boldsymbol{\Pi}_m$$
$$\boldsymbol{H} = i\omega\varepsilon\nabla\times\boldsymbol{\Pi}_m + \nabla\times(\nabla\times\boldsymbol{\Pi}_m)$$

We choose $\boldsymbol{\Pi}=\Pi\boldsymbol{e}_r$ and establish

$$E_r = \frac{\partial^2\Pi_e}{\partial r^2} + k^2\Pi_e$$
$$E_\theta = \frac{1}{r}\frac{\partial^2\Pi_e}{\partial r\partial\theta} - \frac{i\omega\mu}{r\sin\theta}\frac{\partial\Pi_m}{\partial\phi}$$
$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2\Pi_e}{\partial r\partial\phi} + \frac{i\omega\mu}{r}\frac{\partial\Pi_m}{\partial\theta}$$

$$H_r = \frac{\partial^2\Pi_m}{\partial r^2} + k^2\Pi_m$$
$$H_\theta = \frac{i\omega\epsilon}{r\sin\theta}\frac{\partial\Pi_e}{\partial\phi} + \frac{1}{r}\frac{\partial^2\Pi_m}{\partial r\partial\theta}$$
$$H_\phi = -\frac{i\omega\epsilon}{r}\frac{\partial\Pi_e}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2\Pi_m}{\partial r\partial\phi}$$

by using

To be checked by yourself.

$$\nabla u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\boldsymbol{e}_\phi \qquad \nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

$$\nabla\cdot\boldsymbol{a} = \frac{1}{r^2}\frac{\partial\left(r^2 a_r\right)}{dr} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta a_\theta\right)}{d\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{d\phi}$$

$$\nabla\times\boldsymbol{a} = \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta a_\phi\right)}{\partial\theta} - \frac{\partial a_\theta}{\partial\phi}\right]\boldsymbol{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial\left(r a_\phi\right)}{\partial r}\right]\boldsymbol{e}_\theta + \frac{1}{r}\left[\frac{\partial\left(r a_\theta\right)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right]\boldsymbol{e}_\phi$$

Solution of wave equations

- Vector wave functions:

By introducing the Hertz function in the above equations, the EM fields can be expressed as vector wave function in the form like:

$$\boldsymbol{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (A_{mn}\boldsymbol{m}_{mn} + B_{mn}\boldsymbol{n}_{mn})$$

with

$$\boldsymbol{m}_{mn} = \left[imz_n(kr)\pi_n^{|m|}(\cos\theta)\boldsymbol{e}_\theta - z_n(kr)\tau_n^{|m|}(\cos\theta)\boldsymbol{e}_\phi \right] e^{im\phi}$$

$$\boldsymbol{n}_{mn} = \frac{1}{kr}\left[\frac{n(n+1)}{kr}\psi_n(kr)P_n^{|m|}(\cos\theta)\boldsymbol{e}_r + \psi_n'(kr)\tau_n^{|m|}(\cos\theta)\boldsymbol{e}_\theta + im\psi_n'(kr)\pi_n^{|m|}(\cos\theta)\boldsymbol{e}_\phi \right] e^{im\phi}$$

- In classical Lorentz Mie theory – scattering of a plane wave by a spherical particle, we have only terms with $m=\pm1$, so cosine and sine functions as well as Legendre function are used, and the vector wave functions are noted as $\boldsymbol{m}_{o1n}$, $\boldsymbol{m}_{e1n}$, $\boldsymbol{n}_{o1n}$, $\boldsymbol{n}_{e1n}$. *But in beam scattering* $\boldsymbol{m}_{mn}$, $\boldsymbol{n}_{mn}$ ***must be used*** *for the solutions to be completed.*

UNIVERSITÉ DE ROUEN CORIA UMR 6614 CNRS - UNIVERSITE et INSA de ROUEN