Optical diagnostics in fluid mechanics
Metrology of particles

Part 1 : Fundamentals

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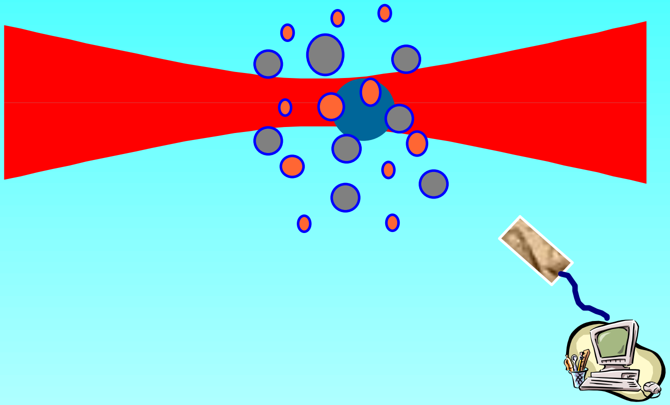
Outline of course



- **Elastic scattering**
- Non-elastic scattering

- **Simple scattering**
- Multiple scattering
- Cohérente scattering

- **Introduction**
- **Fundamentals**
- **Macroscopic models of light scattering by particles**
  - **Approximate models**
  - **Rigorous theories**
  - **Numerical methods**
- **Measurement techniques**
  - LDV, PDA
  - Imaging techniques
  - Refractometry
  - Extinction spectrometry
- **Research topics**

Introduction – macroscopic models

- **Approximate models**
  - Rayleigh: $l << \lambda$,
  - Rayleigh-Gans: $|m - 1| << 1$
  - Diffraction: $l \sim \lambda$
  - GTD - Geometrical Theory of Diffraction,
  - Geometrical optics: $l >> \lambda$
  - **VCRM – Vectorial Complex Ray Model**
- **Rigorous theories**
  - **TLM – Lorenz-Mie Theory**
  - GLMT – Generalized Lorenz-Mie Theory
  - Debye theory / series
- **Numerical methods**
  - T-Matrices,
  - DDA - Dipole discretization Approximation (ADDA, DDSCAT)
  - FDTD - Finite-difference time-domain
  - MoM – Method of Moment
  - FEM – Finite Element Method

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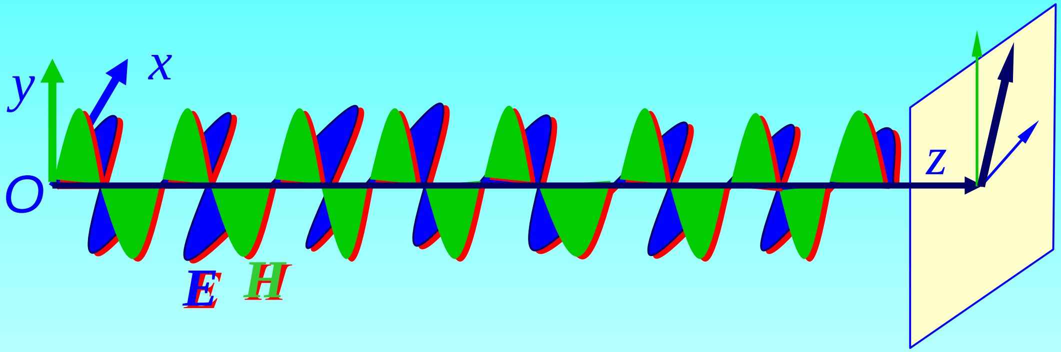
Introduction – Measurement techniques

- **ADL – PDA:**
  - ✓ *velocity, size*
  - ✓ *individual particle*
- **Imaging techniques (holography, PIV, HPIV, …):**
  - ✓ *Size, velocity*
  - ✓ *individual particle or cloud*
- **Rainbow refractometry:**
  - ✓ *Size, refractive index (very accurate)*
  - ✓ *Big particles*
- **Extinction spectrometry**
  - ✓ *Size, concentration*
  - ✓ *Small particles (d~λ)*

Fundamentals – plane wave

Electromagnetic wave (EM) and its properties



Wave front = plans || xOy
Wave vector **k** ⊥ **E** et **H**

Wave vector: $\boldsymbol{k}$

Wave number: $k = \frac{2\pi}{\lambda}$

Two polarisations: $\begin{cases} E_x = A_x \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi_0) \\ E_y = A_y \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi_0) \end{cases}$

$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \vec{k}.\vec{r} - \phi_0)}$ direction of ***E*** → polarization

In a homogeneous and isotropic medium:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H} \qquad \boldsymbol{H} = \frac{1}{\mu\omega}\boldsymbol{k} \wedge \boldsymbol{E}$$

E – electric field
H –magnetic field
ε – permittivity
μ - permeability

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Fundamentals – complex refractive index

Definition and physical interpretation of refractive index

Complex refractive index : $\tilde{m}^2 = \varepsilon$

$$\boxed{\tilde{m} = m_r - m_i i}$$

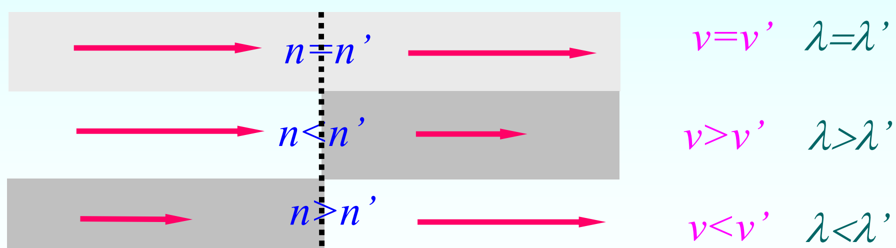
Real part – velocity : $m_r = \dfrac{c}{v}$

Examples:

In the vacuum (air): $c = 3\times10^{8}$ m·s^{-1} , λ=0,6328 µm

n_{water}=1,33 v_{water}=2,26×10^{8} m·s^{-1} , λ_{water}=0,4758 µm

n_{glass}=1,5 v_{glass}=2,00×10^{8} m·s^{-1} , λ_{glass}=0,4219 µm



Fundamentals – complex refractive index

Definition and physical interpretation of refractive index

imaginary part - absorption:

Penetration depth:

$$d = \frac{1}{m_i k_0} = 0.16\frac{\lambda}{m_i}$$

$$E = E_0 e^{i(\omega t - m k_0 z + \phi)}$$
$$= E_0 e^{i\omega t - i m_r k_0 z - m_i k_0 z + i\phi}$$
$$= E_0 e^{-m_i k_0 z} e^{i(\omega t - m_r k_0 z + \phi)}$$

Amplitude à z:

$$E_0(z) = E_0(z=0) e^{-m_i k_0 z}$$

Penetration depth d:

$$\frac{E_0(z=d)}{E_0(z=0)} = e^{-1}$$

$$\Rightarrow d = \frac{1}{m_i k_0} = 0.16\frac{\lambda}{m_i}$$

λ = 0.6328 µm

m_i = 0.1, d = 1 µm

m_i = 0.0001, d = 1 mm

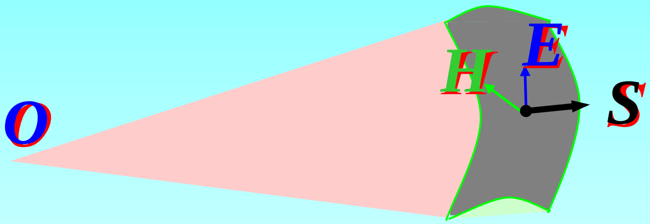
Fundamentals – balance of energy

Energy density and light intensity

Energy density (J/m^3): $$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

Poynting vector (W/m^2):

$$\vec{S} = \vec{E}\wedge\vec{H} = \boxed{\frac{1}{2}\mathrm{Re}\left(\vec{E}\wedge\vec{H}^*\right)}$$



Complex function

In a homogeneous and isotropic medium: $\vec{H} = \frac{1}{\mu\omega}\vec{k}\wedge\vec{E}$

$$\vec{S} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\wedge\vec{H}^*\right) = \frac{1}{2\mu\omega}\mathrm{Re}\left[\vec{E}\wedge\left(\vec{k}\wedge\vec{E}^*\right)\right] = \frac{1}{2\mu\omega}\|\vec{E}\|^2\,\vec{k}$$

In an isotropic medium: $\vec{S} = vu\vec{n}$

Intensity: $I = \left\|\vec{S}\right\| \propto E^2$

Fundamentals – Polarization

Relation between incident wave and scattered wave
In far field

important

$$\begin{pmatrix} E_{\|s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{\|i} \\ E_{\perp i} \end{pmatrix}$$

$S_3 = S_4 = 0$ for homogeneous or multilayered sphere

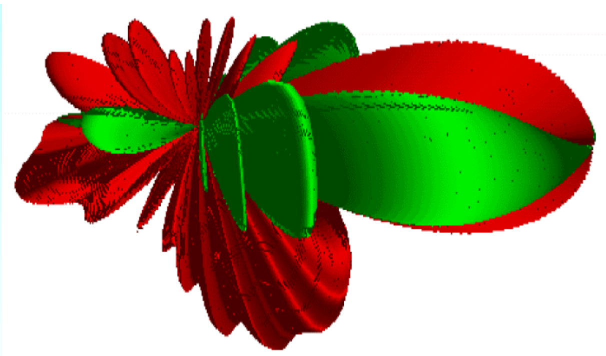
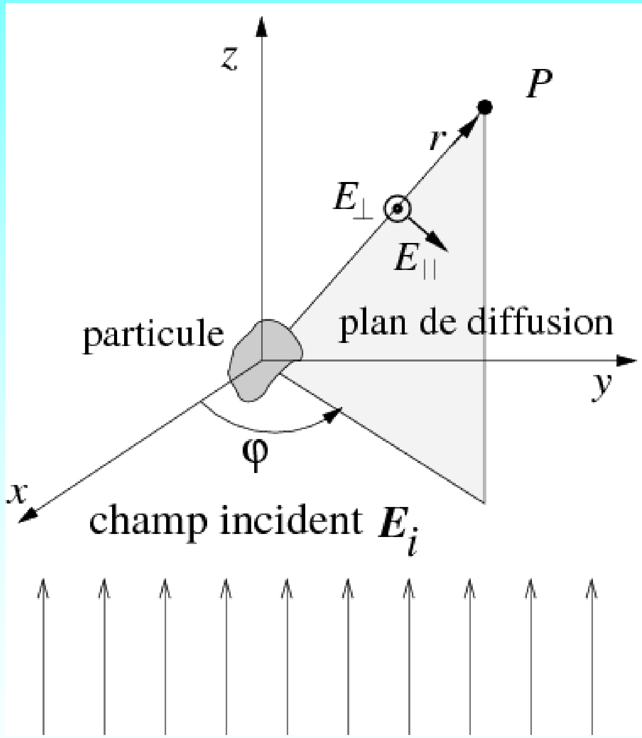


Diagram of a sphere illuminated by a Gaussian beam (off axis), perpendicular polarization in red and parallel in green



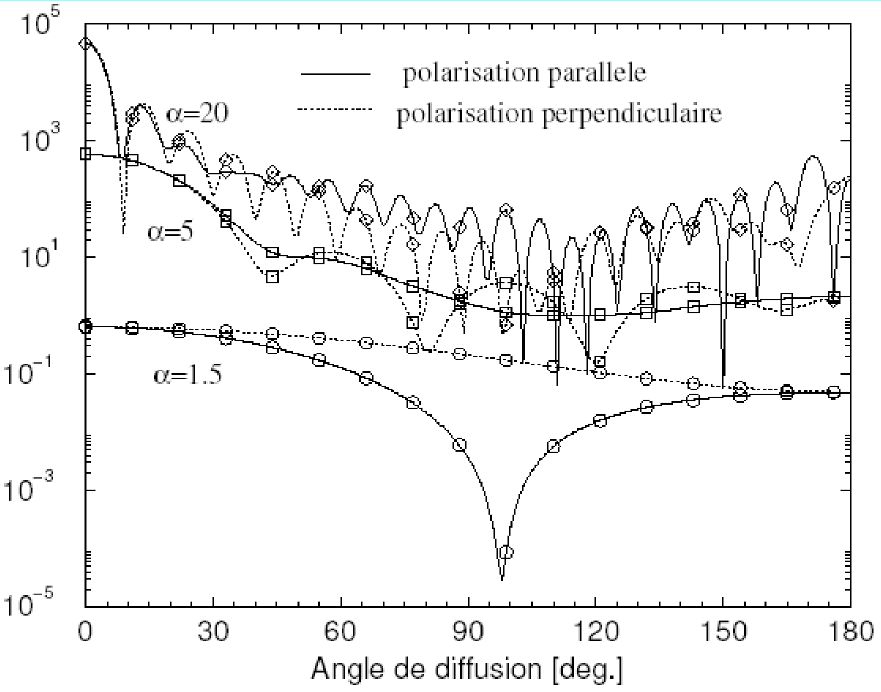
An exercise →

Fundamentals – scattering diagram

Angular distribution of intensity of the scattered wave in far field

A sphere of diameter *d* = αλ/π and refractive index *m* = 1.33 illuminated by a plane wave

important



$$I(r,\theta,\varphi) = \frac{I_0 F(\theta,\varphi)}{k^2 r^2}$$

To be done:
We know S_j (j=1, …, 4) are only function of (θ,φ) in far field and $E_{\perp i} = E_0 p_\perp e^{i\omega t}$, $E_{\parallel i} = E_0 p_\parallel e^{i\omega t}$.
Show this relation with scattering matrix according to $I(r,\theta,\varphi) = ||\mathbf{S}||$.

$$I_{||}(\theta) = F(\theta, \phi = 0) = |S_2|^2$$

$$I_{\perp}(\theta) = F(\theta, \phi = 90°) = |S_1|^2$$

Size parameter : $\alpha = \dfrac{\pi d}{\lambda}$

Fundamentals – Extinction, scattering, absorption

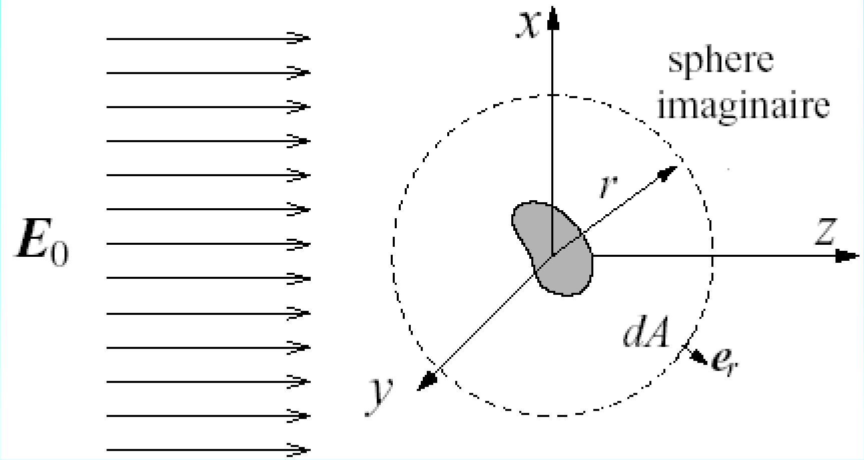
Integral properties of a scattering particle

Outside of the particle: $\boldsymbol{E}=\boldsymbol{E}_i+\boldsymbol{E}_s\,,\qquad \boldsymbol{H}=\boldsymbol{H}_i+\boldsymbol{H}_s$

Poynting vector :

$$\boldsymbol{S}=\frac{1}{2}Re\{\boldsymbol{E}\times\boldsymbol{H}^*\}=\boldsymbol{S}_i+\boldsymbol{S}_s+\boldsymbol{S}_{ext}$$

$$\begin{aligned}\boldsymbol{S}_i &= \frac{1}{2}Re\{\boldsymbol{E}_i\times\boldsymbol{H}_i^*\}\\ \boldsymbol{S}_s &= \frac{1}{2}Re\{\boldsymbol{E}_s\times\boldsymbol{H}_s^*\}\\ \boldsymbol{S}_{ext} &= \frac{1}{2}Re\{\boldsymbol{E}_i\times\boldsymbol{H}_s^*+\boldsymbol{E}_s\times\boldsymbol{H}_i^*\}\end{aligned}$$



Energy balance

$$W_a=-\int_A \boldsymbol{S}\cdot\boldsymbol{e}_r dA=W_i-W_s+W_{ext}$$

$$W_i=-\int_A \boldsymbol{S}_i\cdot\boldsymbol{e}_r dA\,,\quad W_s=\int_A \boldsymbol{S}_s\cdot\boldsymbol{e}_r dA\,,\quad W_{ext}=-\int_A \boldsymbol{S}_{ext}\cdot\boldsymbol{e}_r dA$$

$$\boxed{W_{ext}=W_{abs}+W_{sca}}$$

Fundamentals – Extinction, scattering, absorption

Definition of sections and factors

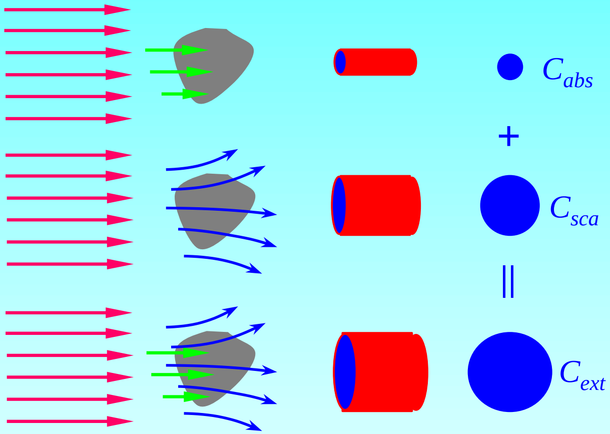
Efficiency sections:

Absorption sections: $C_{abs} = \dfrac{W_{abs}}{I_i}$

Scattering section: $C_{sca} = \dfrac{W_{sca}}{I_i}$

Extinction section : $C_{ext} = \dfrac{W_{ext}}{I_i}$

Physical interpretation



Absorption
+
Scattering
||
total perturbation

A is the geometrical section of the particle **area** projected in the plan ⊥ incident direction.

Efficiency factors

$$Q_{ext} = \frac{C_{ext}}{A}, \quad Q_{abs} = \frac{C_{abs}}{A}, \quad Q_{sca} = \frac{C_{sca}}{A}$$

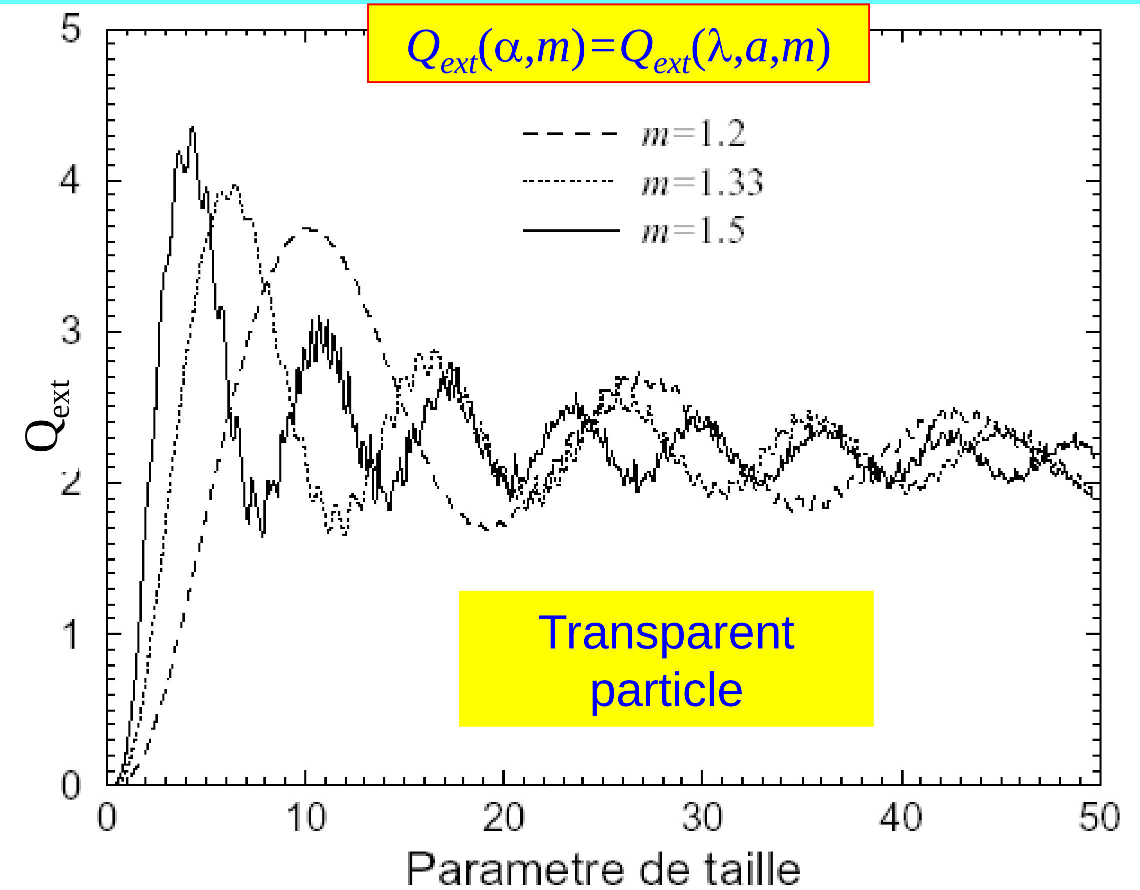
Transparent /no absorbing

$$C_{abs} = 0, \quad C_{ext} = C_{sca}$$

$$Q_{abs} = 0, \quad Q_{ext} = Q_{sca}$$

Fundamentals – Extinction, scattering, absorption

Read and understand the graphics



Small particle

$$Q_{ext} \sim \frac{d^4}{\lambda^4}$$

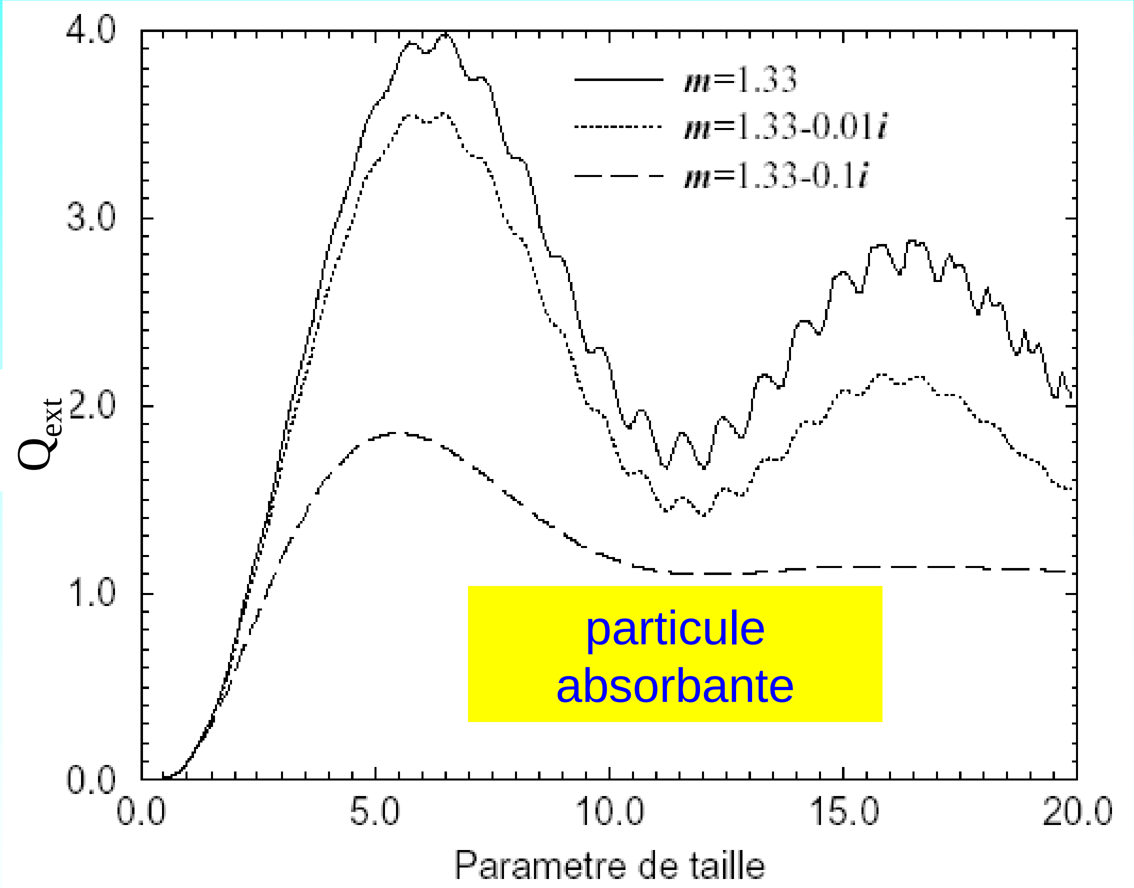
Big particle

$$Q_{ext} \to 2$$

Why is the sky blue and the sun red at the rising and at the set ?

Fundamentals – Extinction, scattering, absorption

Read and understand the graphics



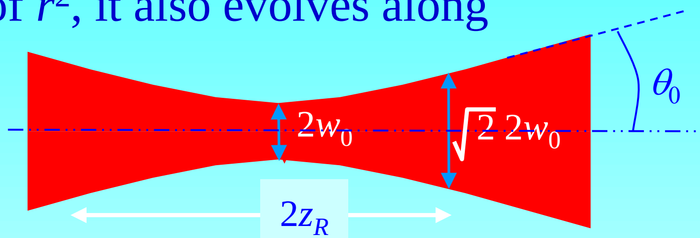
- The greater the absorption, the more smooth is the curve.
- The higher the absorption, the smaller the extinction factor.
- For small particles:
  $Q_{ext} \propto \alpha$
- For big particles:
  $Q_{ext} \rightarrow 1$

i.e.: $C_{ext} \rightarrow A$

Fundamentals – Gaussian beam

Intensity distribution and geometric shape

(a). Intensity decreases exponentially as function of r^2, it also evolves along the beam axis.



$$I(r,z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2r^2}{w^2(z)}\right]$$

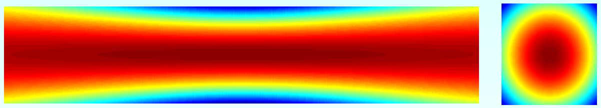
(b). Beam waist radius

$$I(r = w(z)) = \frac{I(r=0,z)}{e^2}$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}$$

(c). Divergence of the beam

$$\theta_0 = \arctan\left(\frac{\lambda}{\pi w_0}\right)$$



(d). Rayleigh length: $z_R = \pi w_0^2 / \lambda$.

$$I(0, z_R) = \frac{I_0}{2}, \qquad w(z_R) = \sqrt{2} w_0$$

Fundamentals – Gaussian beam

Some examples

Divergence:

1. λ= 600 nm :

w_0 = 10 µm: z_R = 500 µm, θ = 0.02 rad

w_0 = 1 mm: z_R = 5 m, θ = 0.0002 rad

w_0 = 1 cm: z_R = 500 m, θ = 0.00002 rad

2. w_0= 100 µm :

λ = 10.6 µm (CO2): z_R = 3 mm, θ = 0.034 rad

λ = 0.6328 µm (He-Ne): z_R = 5 cm, θ = 0.002 rad

λ= 0.488 µm (bleu YAG): z_R = 6.4 cm, θ = 1.5 mrad

Intensity:

At the center:

$$I=I_0$$

On the border of the beam:

$$I(r=w_0)=\frac{I(r=0,z=0)}{e^2}=\frac{I_0}{e^2}$$

Total power of the beam:

$$P=\int I(r,z)dS=\int_0^{2\pi}\int_0^{\infty} I(r,z)rdrd\theta$$

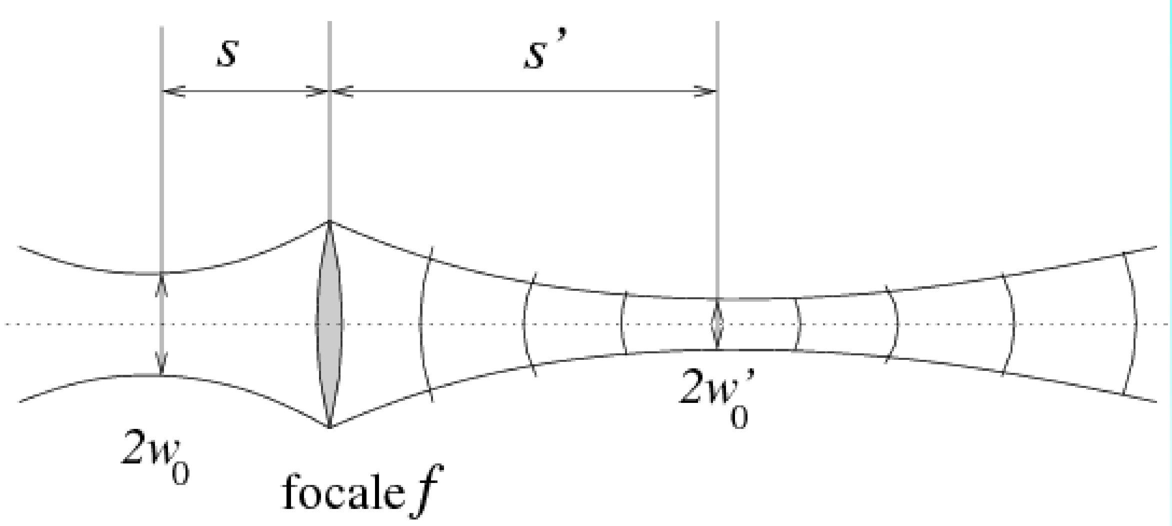
$$=2\pi I_0\left[\frac{w_0}{w(z)}\right]^2\int_0^{\infty}\exp\left[-\frac{2r^2}{w^2(z)}\right]rdr$$

$$=I_0\frac{\pi w_0^2}{2}$$

Fundamentals – Gaussian beam

Thin lens equation

Position of the waists before and after a lens



$$\frac{1}{s'} - \frac{1}{s + \dfrac{z_R^2}{s + f'}} = \frac{1}{f'}$$

or

$$s' = f'\left[1 - \frac{f'(s + f')}{(s + f')^2 + z_R^2}\right]$$

Fundamentals – Gaussian beam

Thin lens equation

Position of the waists before and after a lens

Consequences :

- For quasi-parallel beams, z_R is much greater than all other distances, therefore $s' = f'$, behaviour of the geometrical optics.
- If the beam waist is in the object **focal plane** of a lens, the beam waist is at the image **focal plane** of the lens, since $s = -f \rightarrow s' = f'$. This result is radically different from the geometrical optics.
- The conjugate relation between s and s' depends not only on f' but also on z_R, hence on λ and w_0, which is different from the geometrical optics

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Fundamentals – Gaussian beam

Thin lens equation

Position of the waists before and after a lens

- **Beam waist ratio**

$$m = \frac{w_0'}{w_0} = \left[\left(1+\frac{s}{f'}\right)^2 + \left(\frac{z_R}{f'}\right)^2\right]^{-1/2} \qquad z_R' = m^2 z_R$$

1. In the case s tends to infinity, $m = 0 \rightarrow w' = 0$ (but **NO**! theoretical the limit of $w \sim \lambda/2$)
2. In the case $s = -f'$ (the waist is at the objet focal plan), so $m = f/z_R = f\lambda/\pi w_0^2$

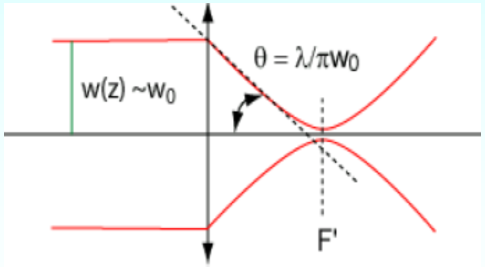
The situation is very different from the GO!

$$\theta' = \lambda/\pi w_0' = w_0/f'.$$

Example: $w_0 = 1$ cm, $f' = 0.1$ m, $\lambda = 0.6328$ µm $\rightarrow z_R = 500$ m, $\theta = 0.001°$

$s = -f' = -0.1$ m $\rightarrow s' = 0.1$ m, $w'_0 = 2$ µm, $\theta' = 5.7°$

We pass from a large waist to a small one …



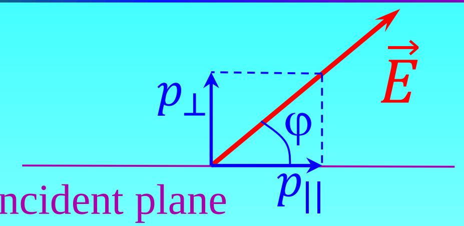
In practice, w_0 can be very small, so that w'_0 is very large.

We find of course all the intermediate cases …

Demonstration *I (r,θ, φ)*:

$$E_{\perp i} = E_0 p_\perp e^{i\omega t},\quad E_{\parallel i} = E_0 p_\parallel e^{i\omega t}$$

$$p_\parallel^2 + p_\perp^2 = 1$$



To simplify the problem, we show for a spherical particle:

$$\boxed{\begin{pmatrix} E_{\parallel s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix} S_2 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} E_{\parallel i} \\ E_{\perp i} \end{pmatrix}}$$

$$\boxed{\vec{E}_s = \frac{e^{ikr-i\omega t}}{-ikr} E_0 \left(S_2 p_\parallel \vec{e}_\parallel + S_1 p_\perp \vec{e}_\perp \right)}$$

$$I = \| \vec{S} \| = \frac{1}{2\mu\omega} \| \vec{E} \|^2 k = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \frac{E_0^2}{(kr)^2} \left[\left(S_2 p_\parallel \right)^2 + \left(S_1 p_\perp \right)^2 \right] = \frac{I_0 F(\theta,\varphi)}{(kr)^2}$$

with

$$I_0 = \frac{E_0^2}{2} \sqrt{\frac{\varepsilon}{\mu}}$$

$$F(\theta,\varphi) = \left(S_2 p_\parallel \right)^2 + \left(S_1 p_\perp \right)^2$$

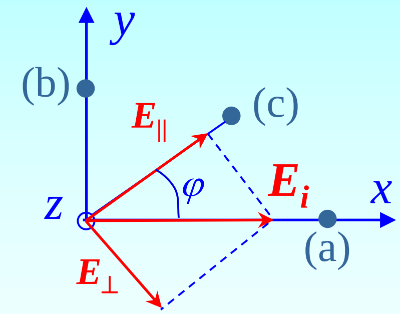
Example of question in exam:

A plane wave of amplitude E_0 polarized in the Oxz plane propagates along the axis Oz. It illuminates a homogeneous spherical particle. Express the scattered field $E_\perp^s$ and $E_\parallel^s$ in the far field as function of the scattering matrix elements S_1, S_2, S_3, S_4 and the observation distance r for the following cases:

(a) the observation point is in the Oxz plane,

(b) the observation point is in the Oyz plane,

(c) the observation point is in the plane containing the Oz axis and makes an angle $\psi = 30°$ relative to the Ox axis.



(a) Parallel polarization: $\boxed{E_\parallel^s = \frac{e^{ikr}}{-ikr} E_0 S_2}$

(b) Perpendicular polarization: $\boxed{E_\perp^s = \frac{e^{ikr}}{-ikr} E_0 S_1}$

(c) 2 components: $\boxed{p_\parallel = \cos 30° = \frac{\sqrt{3}}{2}, \qquad p_\perp = \sin 30° = \frac{1}{2}}$

$$\boxed{E_\parallel^s = \frac{e^{ikr}}{-ikr} E_\parallel^s S_2 = \frac{e^{ikr}}{-ikr} \frac{\sqrt{3}}{2} E_0 S_2}$$

$$\boxed{E_\perp^s = \frac{e^{ikr}}{-ikr} E_\perp^s S_1 = \frac{e^{ikr}}{-ikr} \frac{1}{2} E_0 S_1}$$

Exercise on extionction

A cloud of water drops (refractive index m=1.333) of thickness L=10 mm is illuminated by a light of two wavelengths λ=0.6328 µm and 0.450 µm.

1. We assume that the drops are monodisperse of diameter d=100 nm. What are the extinction coefficients of particles at the two wavelengths?
2. The measured transmittance I/I_0 at λ=0.6328 is 0.8. Deduce the concentration of particles. What is then the transmittance at λ=0.450 µm?
3. Next, we suppose that the environment is composed of two populations of drops with diameters of 100 nm and 1000 nm with a concentration of 10^{17} m^{-3} and 10^{13} m^{-3} respectively. What is the transmittance of the medium?

α_1=0.5, α_2=0.7 $(\alpha_2/\alpha_1)^4$=3.8

1. $d \ll \lambda$, so Rayleigh approximation: $Q_{ext} = \frac{8\alpha^4}{3}\Re\left(\frac{m^2-1}{m^2+2}\right) = 0,0333$, 0.128
2. Beer law: $N = \frac{\ln(I_0/I)}{LC_{ext}} = \frac{\ln(1/0,8)}{10000 \times 0,0333 \times \pi 0,05^2} = 0.0085\ \mu\text{m}^{-3} = 8,5 \times 10^{15}\ \text{m}^{-3}$

   I/I_0=$0.8^{3.8}$=0.42
3. Q_{ext}(100nm)= 0.0333, Q_{ext}(1µm) = 3.6 (see figure)

$$\frac{I}{I_0} = \exp[-L(N_1C_{ext,1} + N_2C_{ext,2})] = 0,58 = 58\%$$

exp[-0.01X10^{17}X$10^{-16}$$\pi$($50^2$x0.033 + 5^2x3.6]
=exp[-0.001X3.14(83+90)]=exp(-0.54)