Lecture at Xidain University
on frontiers of modern optics

Scattering of shaped beam by particles and its applications

II. Scattering of plane wave by regular shape particles

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西安电子科技大学
现代光学前沿专题

波束散射理论和应用

第二讲：规则形状粒子对平面波的散射

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Plan of lecture

- **General description**
- **Scattering by a homogeneous cylinder**
  - Scattering of a plane wave by conductor cylinder
  - Scattering of a plane wave by dielectric cylinder
- **Scattering by a homogeneous sphere**
  - Lorenz-Mie theory and Debye series
- **Some simple but important notes**

General description

General procedure

1. Description of incident EM wave
2. Expansion of the incident beam
3. Expressions of the internal and scattering waves
4. Solving the boundary problem
5. Formula of physical quantities

General description

Wave functions and indices in 4 simple coordinate systems

parameters	Sphere	Spheroid	Cylinder	Elliptical cylinder
coordinates	r, θ, ϕ	ξ, η, ϕ	ρ, θ, z	μ, θ, z
indices	m, n	m, n	n, γ	n, γ
functions	$z_n(r)$, $P_n^m(\theta)$, $e^{im\phi}$	$R_{nm}(\xi)$, $S_{nm}(\eta)$, $e^{im\phi}$	$Z_n(\rho)$, $e^{in\theta}$, $e^{ik\gamma z}$	$seh_n(\mu,q^2)$, $ceh_n(\mu,q^2)$, $se_n(\theta,q^2)$, $ce_n(\theta,q^2)$, $e^{ik\gamma z}$, $q^2=f(\gamma)$
remarks	m, n are integers, indices of wave components; γ is continuous and can be understood as cosine of incident angle in terms of plane wave expansion.			

$z_n(r)$: spherical function, $P_n^m(\theta)$: Associated Legendre function,

$R_{nm}(\xi)$, $S_{nm}(\eta)$: radial and angular spheroidal functions,

$Z_n(\rho)$: cylindrical Bessel function,

$se_n(x)$, $ce_n(x)$ and $seh_n(x)$ $ceh_n(x)$: Mathieu and modified Mathieu function.

General description

The incident, scattered and internal waves are expanded in terms of these functions with corresponding coefficients

	Coefficients	Examples		dependence
		Spherical coordinates	Cylindrical coordinates	
Incident wave	Beam shape coefficients (BSC)	$g_n^m{}_{TM}$, $g_n^m{}_{TE}$	$I_n^{TM}(\gamma)$, $I_n^{TE}(\gamma)$	Independent of particle properties*
Scattered wave	Scattering (Mie) coefficients	a_n, b_n	a_{nI}, a_{nII}, b_{nI},b_{nII}	Independent of incident beam shape;
Internal field	Internal field coefficients	c_n, d_n, (e_n, f_n)	c_{nI}, c_{nII}, d_{nI}, d_{nII}	

* In the case of spheroidal and elliptical cylinder, the aspect ratio intervenes in the beam shape coefficients because of the coordinate property.

Scattering of an infinite cylinder

1. Scattering of a plane wave by a conductor cylinder

I.TM wave: the plane wave propagates in *x* direction and polarized in *xz* plane:

$$\boldsymbol{E}_i = E_{zi}\boldsymbol{e}_z = E_0 e^{-ikx}\boldsymbol{e}_z = E_0 e^{-ik\rho\cos\phi}\boldsymbol{e}_z$$

•According to the generation function of Bessel function, the incident wave can be expanded as

$$E_{zi} = E_0 e^{-ik\rho\cos\phi} = E_0 \sum_{m=-\infty}^{+\infty} i^{-m} J_m(k\rho) e^{im\phi}$$

•Suppose that the scattered field is expanded as (since $H_m^{(2)}(k\rho) \xrightarrow[\rho\to\infty]{} e^{-ik\rho}/\sqrt{k\rho}$)

$$E_{zs} = E_0 \sum_{m=-\infty}^{+\infty} i^{-m} a_m H_m^{(2)}(k\rho) e^{im\phi}$$

Can you resolve this simple question?

•The total electric field on the surface of the cylinder $\rho = a$ is null

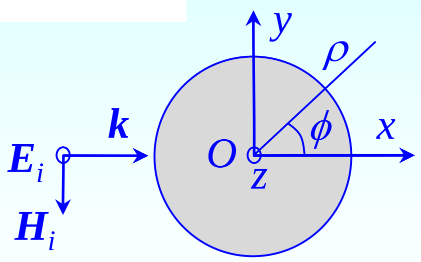
$$E_z = E_{zi} + E_{zs} = E_0 \sum_{m=-\infty}^{+\infty} i^{-m} \left[J_m(ka) + a_m H_m^{(2)}(ka) \right] e^{im\phi} = 0$$

•So the scattering coefficients:

$$a_m = -\frac{J_m(ka)}{H_m^{(2)}(ka)}$$

•The total magnetic field is obtained by

$$\boldsymbol{H} = \frac{i}{\mu\omega} \nabla \times \boldsymbol{E}$$



Scattering of an infinite cylinder

1. Scattering of a plane wave by a conductor cylinder

II.TE wave: the plane wave propagates in *x* direction and polarized in *xy* plane:

$$\boldsymbol{H}_i = H_{zi}\boldsymbol{e}_z = H_0 e^{-ikx}\boldsymbol{e}_z = H_0 e^{-ik\rho\cos\phi}\boldsymbol{e}_z$$

•According to the generation function of Bessel function, the incident wave can be expanded as

$$H_{zi} = H_0 e^{-ik\rho\cos\phi} = H_0 \sum_{m=-\infty}^{+\infty} i^{-m} J_m(k\rho) e^{im\phi}$$

Try for another polarization?

•Suppose that the scattered field is expanded as

$$H_{zs} = H_0 \sum_{m=-\infty}^{+\infty} i^{-m} b_m H_m^{(2)}(k\rho) e^{im\phi}$$

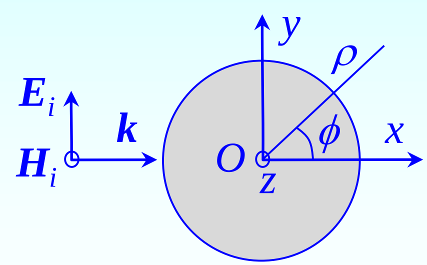
•The tangent component of the electric field is null: E_ϕ=0. We know al $\boldsymbol{E} = -\frac{i}{\varepsilon\omega}\nabla\times\boldsymbol{H}$

$$E_\phi = \frac{iH_0}{\varepsilon\omega}\sum_{m=-\infty}^{+\infty} i^{-m}\left[J_m'(ka) + b_m H_m^{(2)'}(ka)\right]e^{im\phi} = 0$$

$$\nabla\times\boldsymbol{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\phi}{\partial z}\right]\boldsymbol{e}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\boldsymbol{e}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right]\boldsymbol{e}_z$$

•So the scattering coefficients:

$$b_m = -\frac{J_m'(ka)}{H_m^{(2)'}(ka)}$$



•The total electric field is obtained by

$$\boldsymbol{E} = -\frac{i}{\varepsilon\omega}\nabla\times\boldsymbol{H}$$

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Scattering of an infinite cylinder

2. Scattering of a plane wave by a conductor cylinder

Summary:

•The total field outside of the cylinder:

TM wave:

$$a_m = -\frac{J_m(ka)}{H_m^{(2)}(ka)}$$

$$\boldsymbol{E}(\rho,\phi,z) = E_z\boldsymbol{e}_z = E_0 \sum_{m=-\infty}^{+\infty} i^{-m}\left[J_m(k\rho) - a_m H_m^{(2)}(k\rho)\right] e^{im\phi}\boldsymbol{e}_z$$

$$\boldsymbol{H}(\rho,\phi,z) = \frac{i}{\mu\omega}\nabla\times\boldsymbol{E}$$

$$= \frac{iE_0}{\mu\omega}\sum_{m=-\infty}^{+\infty} i^{-m}\left\{\frac{im}{k\rho}\left[J_m(k\rho) - a_m H_m^{(2)}(k\rho)\right]\boldsymbol{e}_\phi + \left[J_m'(k\rho) - a_m H_m^{(2)'}(k\rho)\right]\boldsymbol{e}_\rho\right\} e^{im\phi}$$

TE wave:

$$b_m = -\frac{J_m'(ka)}{H_m^{(2)'}(ka)}$$

$$\boldsymbol{E}(\rho,\phi,z) = \frac{1}{i\varepsilon\omega}\nabla\times\boldsymbol{H}$$

$$= \frac{H_0}{i\varepsilon\omega}\sum_{m=-\infty}^{+\infty} i^{-m}\left\{\frac{im}{k\rho}\left[J_m(k\rho) - b_m H_m^{(2)}(k\rho)\right]\boldsymbol{e}_\phi + \left[J_m(k\rho) - b_m H_m^{(2)}(k\rho)\right]\boldsymbol{e}_\rho\right\} e^{im\phi}$$

$$\boldsymbol{H}(\rho,\phi,z) = E_z\boldsymbol{e}_z = H_0 \sum_{m=-\infty}^{+\infty} i^{-m}\left[J_m(k\rho) - b_m H_m^{(2)}(k\rho)\right] e^{im\phi}\boldsymbol{e}_z$$

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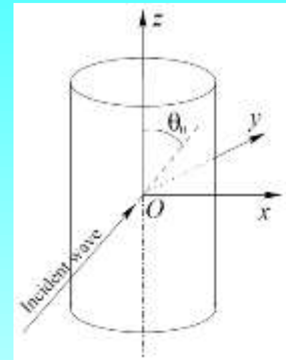
Scattering of an infinite cylinder

2. Scattering of a plane wave by a dielectric cylinder

I.TM wave: the plane wave propagates and polarized in *xz* plane, ***k*** makes an angle θ_0 with *z* axis.



•Incident wave:

h=$k\cos\theta_0$,
μ=$k\sin\theta_0$

$$\boldsymbol{E}_i = E_0(-\cos\theta_0\boldsymbol{e}_x + \sin\theta_0\boldsymbol{e}_z)e^{-ik(x\sin\theta_0 + z\cos\theta_0)}$$
$$= \frac{E_0}{k}(-h\cos\phi\boldsymbol{e}_\rho + h\sin\phi\boldsymbol{e}_\phi + \text{礶}\boldsymbol{e}_z) \quad {}^{-i(\text{礹}\cos\phi + \ z)}$$

•Expansion of the incident wave:

Bessel generation function: $f(\text{礶},\phi) = \quad {}^{-i\text{祄}\cos\phi} = \sum_{m=-\infty}^{\infty} i^{-} \ J_m(\text{礶}) \quad {}^{im\phi}$

$$\sin\phi e^{-i\text{祄}\cos\phi} = \frac{-i}{k\rho}\frac{\partial f}{\partial\phi} = \frac{m}{?\rho}\sum_{m=-\infty}^{\infty} i^{-} \ J_m(k\rho)e^{im\phi}$$

$$\cos\phi e^{-i\text{祄}\cos\phi} = i\frac{\partial f}{\partial(\mu\rho)} = i\sum_{m=-\infty}^{\infty} i^{-} \ J_m{}'(k\rho)e^{im\phi}$$

We obtain

$$\boldsymbol{E}_i = E_0\sum_{m=-\infty}^{\infty} E_m\left[-ihJ_m^{'}(\text{礹})\boldsymbol{e}_\rho + \frac{hm}{\mu\rho} \ {}_m(\text{梣})\boldsymbol{e}_\phi + \ J_m(\text{礶})\boldsymbol{e}_z\right] {}^{i(m\phi - hz)}$$

with $E_m = \dfrac{i^{-m}}{k}$. This can be written in vector wave function:

$$\boldsymbol{E}_i = E_0\sum_{m=-\infty}^{\infty} E_m\boldsymbol{n}_m$$

> E_m is different from Bohren.
> Better to use k and without E_0.

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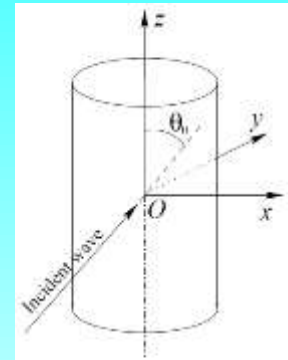
Scattering of an infinite cylinder

2. Scattering of a plane wave by a dielectric cylinder

•The incident TM fields are expanded as

$$\boldsymbol{E}_i = E_0 \sum_{m=-\infty}^{\infty} E_m \boldsymbol{n}_m \qquad \boldsymbol{H}_i = -E_0 \frac{ik}{\mu\omega} \sum_{m=-\infty}^{\infty} E_m \boldsymbol{m}_m$$



•The internal and external fields are expanded as

$$\boldsymbol{E}_1 = E_0 \sum_{m=-\infty}^{\infty} E_m \left[c_{mI} \boldsymbol{m}_m + d_{mI} \boldsymbol{n}_m\right]$$

$$\boldsymbol{H}_1 = E_0 \frac{-ik_1}{\omega\mu} \sum_{m=-\infty}^{\infty} E_m \left[c_{mI} \boldsymbol{n}_m + d_{mI} \boldsymbol{m}_m\right]$$

$$H_m^{(2)}(\rho) \xrightarrow{\rho \gg n} \sqrt{\frac{2}{\pi\rho}} e^{-i\rho} i^m e^{\pi/4}$$

$$\boldsymbol{E}_s = -E_0 \sum_{m=-\infty}^{\infty} E_m \left[ia_{mI} \boldsymbol{m}_m^{(4)} + b_{mI} \boldsymbol{n}_m^{(4)}\right]$$

$$\boldsymbol{H}_s = E_0 \frac{ik}{\omega\mu} \sum_{m=-\infty}^{\infty} E_m \left[ia_{mI} \boldsymbol{n}_m^{(4)} + b_{mI} \boldsymbol{m}_m^{(4)}\right]$$

•By using the boundary conditions for electric and magnetic waves at $\rho = a$:

$$E_{i\phi} + E_{s\phi} = E_{1\phi} \qquad H_{i\phi} + H_{s\phi} = H_{1\phi}$$

$$E_{iz} + E_{sz} = E_{1z} \qquad H_{iz} + H_{sz} = H_{1z}$$

Scattering of an infinite cylinder

2. Scattering of a plane wave by a dielectric cylinder

•The coefficients are found: $\xi = \alpha \sin\theta_0, \quad \eta = \alpha\sqrt{m^2 - \cos^2\theta_0}, \quad \alpha = ka$

$$\begin{aligned} a_{mI} &= \frac{C_m V_m - B_m D_m}{W_m V_m + iD_m^2} \\ b_{mI} &= \frac{W_m B_m + iD_m C_m}{W_m V_m + iD_m^2} \\ a_{mII} &= \frac{iC_m D_m - A_m V_m}{W_m V_m + iD_m^2} \\ b_{mII} &= -i\frac{C_m W_m + A_m D_m}{W_m V_m + iD_m^2} \end{aligned}$$

with

$$\begin{aligned} A_m &= i\xi[\xi J_m'(\eta)J_m(\xi) - \eta J_m(\eta)J_m'(\xi)] \\ B_m &= \xi[m^2\xi J_m'(\eta)J_m(\xi) - \eta J_m(\eta)J_m'(\xi)] \\ C_m &= n\cos\theta_0 \eta J_m(\eta)J_m(\xi)(\frac{\xi^2}{\eta^2} - 1) \\ D_m &= n\cos\theta_0 \eta J_m(\eta)H_m^{(1)}(\xi)(\frac{\xi^2}{\eta^2} - 1) \\ V_m &= \xi[m^2\xi J_m'(\eta)H_m^{(1)}(\xi) - \eta J_m(\eta)H_m^{(1)'}(\xi)] \\ W_m &= i\xi[\eta J_m(\eta)H_m^{(1)'}(\xi) - \xi J_m'(\eta)H_m^{(1)}(\xi)] \end{aligned}$$

•Special case of normal incidence: $\theta_0 = 90°, \quad \xi = \alpha, \quad \eta = \alpha m$

$$C_m = D_m = 0 \quad \Rightarrow \quad a_{mI} = b_{mII} = 0$$

$$b_{mI} = \frac{B_m}{V_m}, \qquad a_{mII} = -\frac{A_m}{W_m}$$

No cross polarization.

•Special case of conductor cylinder m:

$$m \to \infty, \qquad \eta = \alpha m \to \infty,$$

$$b_{mI} = \frac{J_m(\xi)}{H_m^{(1)}(\xi)}, \qquad a_{mII} = \frac{J_m'(\xi)}{H_m^{(1)'}(\xi)}$$

Identical to our previous results.

Scattering of a homogeneous sphere

1. Lorenz-Mie theory

Scattering of the plane wave by a homogeneous sphere

- **Incident wave:**

$$\boldsymbol{E}_i = E_0 e^{-ikz} \boldsymbol{e}_x = E_0 e^{-ikr\cos\theta} \boldsymbol{e}_x$$

cf. Bohren Chapt.4 王一平，工程电动力学

The incident wave is polarized in x direction.

- **Vector wave functions:**

$$\boldsymbol{m}_{\substack{e\\o}1n} = \mp z_n(kr)\pi_n(\cos\theta) \begin{matrix}\sin\phi\\ \cos\phi\end{matrix} \vec{e}_\theta - z_n(kr)\tau_n(\cos\theta) \begin{matrix}\cos\phi\\ \sin\phi\end{matrix} \vec{e}_\phi$$

$$\boldsymbol{n}_{\substack{e\\o}1n} = \frac{1}{kr}\left[n(n+1) z_n(kr) P_n(\cos\theta) \begin{matrix}\cos\phi\\ \sin\phi\end{matrix} \vec{e}_r + [krz_n(kr)]' \tau_n(\cos\theta) \begin{matrix}\cos\phi\\ \sin\phi\end{matrix} \vec{e}_\theta \mp [krz_n(kr)]' \pi_n(\cos\theta) \begin{matrix}\sin\phi\\ \cos\phi\end{matrix} \vec{e}_\phi \right]$$

- **Expansion of incident wave :**

$$\boldsymbol{E}_i = E_0 \sum_{n=0}^{\infty} \left[c_n^{pw} \boldsymbol{m}_{o1n}^{(1)} + d_n^{pw} n_{e1n}^{(1)} \right]$$

- **Internal and scattered fields:**

$$\vec{E}_s = E_0 \sum_{n=1}^{\infty} C_n^{pw} (i a_n \vec{n}_{e1n}^{(4)} - b_n \vec{m}_{o1n}^{(4)})$$

$$\vec{E}_e = E_0 \sum_{n=1}^{\infty} C_n^{pw} (c_n \vec{m}_{o1n}^{(1)} - i d_n \vec{n}_{e1n}^{(1)})$$

Scattering of a homogeneous sphere

- **Expansion of the incident wave:** $\boldsymbol{E}_i = E_0 e^{ikz}\boldsymbol{e}_x = E_0 e^{ikr\cos\theta}\boldsymbol{e}_x$
  - Scalar potential:

$$\Pi_e^i = \cos\phi \sum_{n=1}^{\infty} c_n^{pw} P(\cos\theta)\, j_n(kr)$$

$$\Pi_m^i = \sin\phi \sum_{n=1}^{\infty} c_n^{pw} P(\cos\theta)\, j_n(kr)$$

with $$c_n^{pw} = i^{-n}\frac{2n+1}{n(n+1)}$$

  - EM fields:

$$\boldsymbol{E}_i = E_0 \sum_{n=1}^{\infty} c_n^{pw}\left[\boldsymbol{m}_{o1n}^{(1)} - i\boldsymbol{n}_{e1n}^{(1)}\right]$$

$$\boldsymbol{H}_i = E_0 \sum_{n=1}^{\infty} c_n^{pw}\left[\boldsymbol{m}_{e1n}^{(1)} + i\boldsymbol{n}_{o1n}^{(1)}\right]$$

Scattering of a homogeneous sphere

- **Expansion of the scattered waves:**
  - Scalar potential :

$$\Pi_e^s = \cos\phi \sum_{n=1}^{\infty} a_n P(\cos\theta) h_n^{(2)}(kr)$$

$$\Pi_m^s = \sin\phi \sum_{n=1}^{\infty} b_n P(\cos\theta) h_n^{(2)}(kr)$$

  - Electric Field:

$$\boldsymbol{E}_s = E_0 \sum_{n=1}^{\infty} c_n^{pw} \left[i a_n \boldsymbol{n}_{e1n}^{(4)} - b_n \boldsymbol{m}_{o1n}^{(4)} \right]$$

- **Expansion of the internal waves:**
  - Scalar potential :

$$\Pi_e^e = \cos\phi \sum_{n=1}^{\infty} c_n P(\cos\theta) j_n(kr)$$

$$\Pi_m^e = \sin\phi \sum_{n=1}^{\infty} d_n P(\cos\theta) j_n(kr)$$

  - Electric Field:

$$\boldsymbol{E}_e = E_0 \sum_{n=1}^{\infty} c_n^{pw} \left[c_n \boldsymbol{m}_{o1n}^{(1)} - i d_n \boldsymbol{n}_{e1n}^{(1)} \right]$$

Scattering of a homogeneous sphere

- **Boundary conditions at *r* = *a*:**

$$E_{i\phi}+E_{s\phi}=E_{e\phi} \qquad H_{i\phi}+H_{s\phi}=H_{e\phi}$$
$$E_{i\theta}+E_{s\theta}=E_{e\theta} \qquad H_{i\theta}+H_{s\theta}=H_{e\theta}$$

- We establish

$$\psi(x)=xj_n(x)$$
$$\xi(x)=xh_n^{(2)}(x)$$
$$\alpha=ka$$

$$\psi_n(\tilde m\alpha)c_n+\tilde m\xi_n(\alpha)b_n=\tilde m\psi_n(\alpha)$$
$$\psi_n'(\tilde m\alpha)c_n+\xi_n'(\alpha)b_n=\psi_n'(\alpha)$$
$$\tilde m\psi_n(\tilde m\alpha)d_n+\xi_n(\alpha)a_n=\psi_n(\alpha)$$
$$\psi_n'(\tilde m\alpha)d_n+\tilde m\xi_n'(\alpha)a_n=\tilde m\psi_n'$$

and find

$$a_n=\frac{\tilde m\psi_n(\tilde m\alpha)\psi_n'(\alpha)-\psi_n(\alpha)\psi_n'(\tilde m\alpha)}{\tilde m\psi_n(\tilde m\alpha)\xi_n'(\alpha)-\xi_n(\alpha)\psi_n'(\tilde m\alpha)}$$

$$b_n=\frac{\psi_n(\tilde m\alpha)\psi_n'(\alpha)-\tilde m\psi_n(\alpha)\psi_n'(\tilde m\alpha)}{\psi_n(\tilde m\alpha)\xi_n'(\alpha)-\tilde m\xi_n(\alpha)\psi_n'(\tilde m\alpha)}$$

Scattering of a homogeneous sphere

- **Resonance:**
  Coefficient of Mie for external field:

$$a_n = \frac{\tilde{m}\psi_n(\tilde{m}\alpha)\psi_n'(\alpha) - \psi_n(\alpha)\psi_n'(\tilde{m}\alpha)}{\tilde{m}\psi_n(\tilde{m}\alpha)\xi_n'(\alpha) - \xi_n(\alpha)\psi_n'(\tilde{m}\alpha)} \qquad b_n = \frac{\psi_n(\tilde{m}\alpha)\psi_n'(\alpha) - \tilde{m}\psi_n(\alpha)\psi_n'(\tilde{m}\alpha)}{\psi_n(\tilde{m}\alpha)\xi_n'(\alpha) - \tilde{m}\xi_n(\alpha)\psi_n'(\tilde{m}\alpha)}$$

Resonance electric:

$$a_n \to \infty \text{ when } \tilde{m}\psi_n(\tilde{m}\alpha)\xi_n'(\alpha) = \xi_n(\alpha)\psi_n'(\tilde{m}\alpha)$$

Resonance magnetic:

$$b_n \to \infty \text{ when } \psi_n(\tilde{m}\alpha)\xi_n'(\alpha) = \tilde{m}\xi_n(\alpha)\psi_n'(\tilde{m}\alpha)$$

Given m to find x
Or given x find m

E, Q_{ext}, P_r. … → ∞

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Scattering of a homogeneous sphere

- **Calculation of physical quantities**
  - **Far field:**

$$E_{s\theta} = iE_0 \frac{e^{-ikr}}{kr} \cos\phi \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \tau_n(\theta) + b_n \pi_n(\theta) \right]$$

$$E_{s\phi} = -iE_0 \frac{e^{-ikr}}{kr} \sin\phi \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \pi_n(\theta) + b_n \tau_n(\theta) \right]$$

  - **Cross sections:**

$$C_{sca} = \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} (2n+1) \left(|a_n|^2 + |b_n|^2 \right)$$

$$C_{ext} = \frac{\lambda^2}{\pi} \mathrm{Re} \sum_{n=1}^{\infty} (2n+1) \mathrm{Re}\left(a_n + b_n \right)$$

  - **Radiation pressure:**

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} |g_n|^2 \mathrm{Re}(a_n + b_n - 2a_n b_n^*) + \frac{n(n+2)}{n+1} \mathrm{Re}[g_n g_{n+1}^* (a_n + b_n + a_{n+1}^* + b_{n+1}^* - 2a_n a_{n+1}^* - 2b_n b_{n+1}^*)].$$

Interpretation

1. Scattering diagram:

$$I = \begin{pmatrix} \cos^2\varphi \\ \sin^2\varphi \end{pmatrix} f(\theta)$$

$$I_{\parallel} = \cos^2\varphi\, f(\theta)$$

$$I_{\perp} = \sin^2\varphi\, f(\theta)$$

2. Resonance:

when a_n or $b_n \to \infty$

- $E \to \infty$
- C_{ext}, $C_{ext} \to \infty$
- $C_{pr,z} \to \infty$

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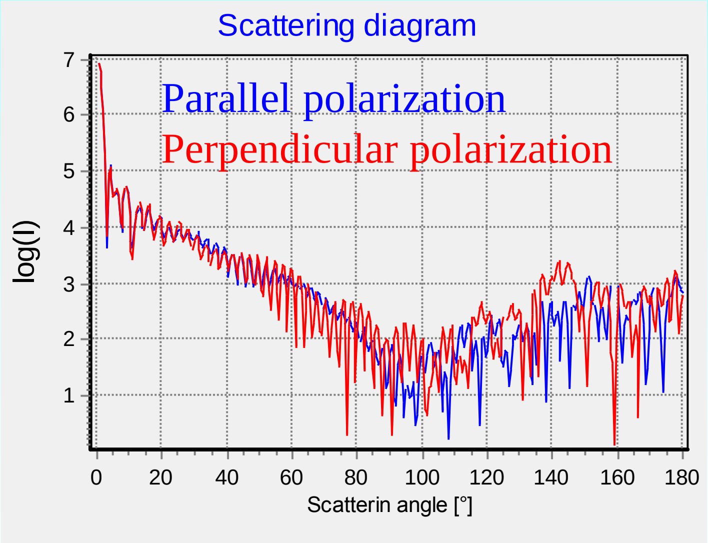
Scattering of a homogeneous sphere

- **Examples of calculation**

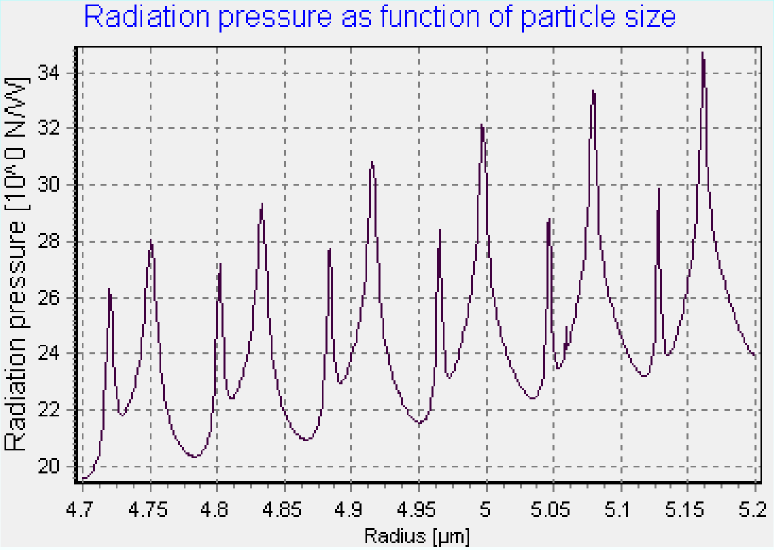
Scattering diagrams

Scattering diagrams of a water droplet (a=7.5 μm m=1.333) illuminated by a plane wave (λ=0.6328 μm).



Radiation pressure

z component of radiation pressure of a water (m=1.333) droplet illuminated by a plane wave (λ=0.6328 μm).



Scattering of a homogeneous sphere

2. Lorenz-Mie theory

Scattering of the plane wave by a conductor sphere

An exercise:

- **As a special case of the Mie theory**
  - **Find** a_n and b_n**:**

$$a_n = \frac{\tilde{m}\psi_n(\tilde{m}\alpha)\psi_n'(\alpha) - \psi_n(\alpha)\psi_n'(\tilde{m}\alpha)}{\tilde{m}\psi_n(\tilde{m}\alpha)\xi_n'(\alpha) - \xi_n(\alpha)\psi_n'(\tilde{m}\alpha)} \qquad b_n = \frac{\psi_n(\tilde{m}\alpha)\psi_n'(\alpha) - \tilde{m}\psi_n(\alpha)\psi_n'(\tilde{m}\alpha)}{\psi_n(\tilde{m}\alpha)\xi_n'(\alpha) - \tilde{m}\xi_n(\alpha)\psi_n'(\tilde{m}\alpha)}$$

$$\text{conductor}: \tilde{m} \to \infty \qquad a_n = \frac{\psi_n'(x)}{\xi_n'(x)}, \qquad b_n = \frac{\psi_n(x)}{\xi_n(x)}$$

- **Direct solution:**
  - Incident wave: $\boldsymbol{E}_i = E_0 \sum_{n=1}^{\infty} c_n^{pw} \left[\boldsymbol{m}_{o1n}^{(1)} - i\boldsymbol{n}_{e1n}^{(1)} \right]$
  - Scattered wave: $\boldsymbol{E}_s = E_0 \sum_{n=1}^{\infty} c_n^{pw} \left[i a_n \boldsymbol{n}_{e1n}^{(4)} - b_n \boldsymbol{m}_{o1n}^{(4)} \right]$
  - Boundary conditions: $E_\phi = E_\theta = 0$. It is ready to find the same a_n and b_n

Scattering of a homogeneous sphere

3. Debye theory

Expansion of the scattered wave in different orders

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{x\psi_n(\beta)\psi'_n(\alpha) - y\psi_n(\alpha)\psi'_n(\beta)}{x\psi_n(\beta)\xi'_n(\alpha) - y\xi_n(\alpha)\psi'_n(\beta)} = \frac{1}{2}\left[1 - R_n^{212} - T_n^{12}T_n^{21}\sum_{p=1}^{\infty}\left(R_n^{121}\right)^{p-1}\right]$$

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \frac{T_n^{21}}{2\left(1 - R_n^{121}\right)} = \frac{1}{2}T_n^{21}\sum_{p=1}^{\infty}\left(R_n^{121}\right)^{p-1}$$

$$R_n^{121} = -\frac{\alpha\xi'_n(x)\xi_n(y) - \beta\xi_n(x)\xi'_n(y)}{\alpha\xi'_n(x)\zeta_n(y) - \beta\xi_n(x)\zeta'_n(y)} = -\frac{\xi_n(y)}{\zeta_n(y)}\frac{\alpha D_n^{(1)}(x) - \beta D_n^{(1)}(y)}{\alpha D_n^{(1)}(x) - \beta D_n^{(2)}(y)}$$

$$R_n^{212} = -\frac{\alpha\zeta'_n(x)\zeta_n(y) - \beta\zeta_n(x)\zeta'_n(y)}{\alpha\xi'_n(x)\zeta_n(y) - \beta\xi_n(x)\zeta'_n(y)} = -\frac{\zeta_n(x)}{\xi_n(x)}\frac{\alpha D_n^{(2)}(x) - \beta D_n^{(2)}(y)}{\alpha D_n^{(1)}(x) - \beta D_n^{(2)}(y)}$$

$$T_n^{12} = \frac{2i}{\alpha\xi'_n(x)\zeta_n(y) - \beta\xi_n(x)\zeta'_n(y)} \qquad T_n^{21} = \frac{m_1}{m_2}\frac{2i}{\alpha\xi'_n(x)\zeta_n(y) - \beta\xi_n(x)\zeta'_n(y)}$$

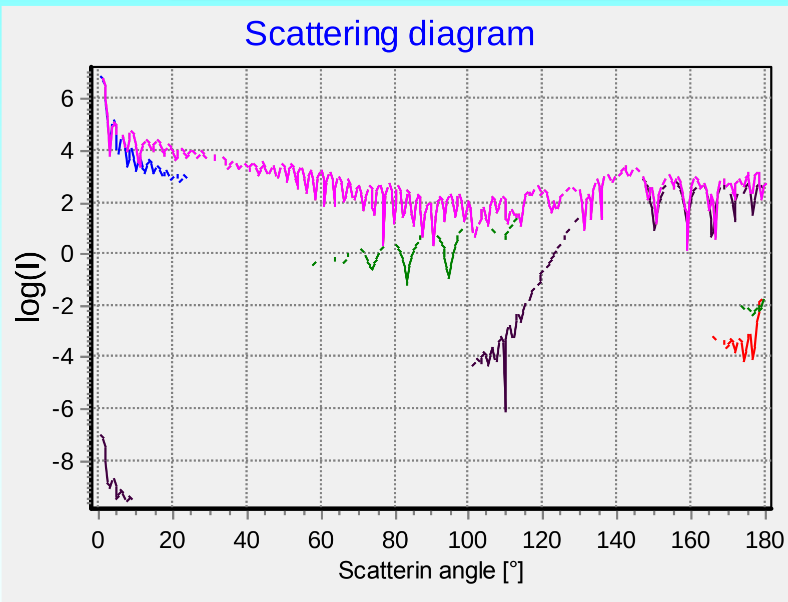
$$\alpha = ka, \quad \beta = m_1 ka, \quad x = \begin{cases} m_1/\mathrm{m}_2 & \text{pour } a_n \\ 1 & \text{pour } b_n \end{cases}, \quad y = \begin{cases} 1 & \text{pour } a_n \\ m_1/\mathrm{m}_2 & \text{pour } b_n \end{cases}$$

Scattering of a homogeneous sphere

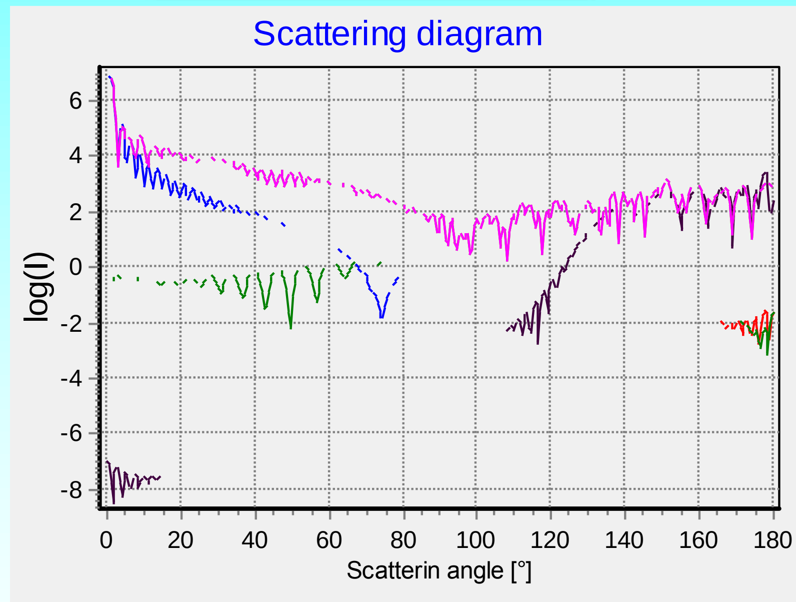
- **Examples of calculation: scattering diagrams**

Scattering diagrams of different orders for a water droplet (a=7.5 μm, m=1.333) illuminated by a plane wave (λ=0.6328 μm).

Perpendicular polarization



Parallel polarization



Total intensity $p = -1,0$: diffraction + reflection. $p = 1$: reflection.

$p=2$: one internal reflection. $p=3$: two internal reflections.

Some simple but important notes

- **Time convention**
  - Two conventions are used in this lecture according to the context, so the attention must be paid when using the formula.
  - The signs of the time and propagation factors are always opposed.

e^{ikr} or e^{ikz} for $e^{-i\omega t}$ e^{-ikr} or e^{-ikz} for $e^{i\omega t}$

  - Relation between the two conventions: all formula for one convention can be converted for the other by replacing i by $-i$

- **Sign of the imaginary part of refractive index**

The sign of the imaginary part of refractive index is always opposed to the sign of time convention (before $i\omega t$)

- **Wavelength in Mie theory**

When the refractive index of the surrounding medium is different from 1 and the *relative* refractive index is used, the wavelength **MUST** be that in the *surrounding* medium, but not the wavelength in the vacuum.

Some simple but important notes

▪ Polarization

Different definitions and nominations are used according to the context.

- **Perpendicular and parallel polarizations:**
  - **Usage:** often for reflection and refraction, scattering wave in far field.
  - **Definition:** electric field relative to the scattering plane (defined by the direction of the incident wave and the observation point).
  - **Exception**: For infinite cylinder, the polarization of the incident wave is defined according to the plane of the cylinder and the incident wave, **but** the scattered field is defined according to the scattering plane (defined by the incident direction and the observation point) (Bohren).
- **Horizontal and vertical polarizations :**
  - **Usage:** mainly in wave propagation, antenna, and microwave, .
  - **Definition:** (Theory of irospheric wave, KC Yeh and CHLiu 1972, 电离层波理论，叶公杰、刘兆汉1983)

Horizontal: ***E*** perpendicular to the incident plane (since surface horizontal).

Vertical = ***H*** perpendicular to the incident plane.

Some simple but important notes

- **TM and TE :**
  - Mean only transversal electric and magnetic field.

    Ex. In GLMT $g_{nm,TE}$ calculated by H_r(E_r,=0), $g_{nm,TM}$ calculated by E_r (H_r,=0).
  - Used for reflection and refraction:(王一平：工程电动力学)

    TE= ⊥ since the polarization of *E* does not change at all for all the waves.

    TM=|| since the polarization of *H* does not change at all for all the waves.