

Lecture at Xidian University
on Frontiers in modern optics

Scattering of shaped beam by particles and its applications

I. Fundamentals of light scattering by small particles

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西安电子科技大学
现代光学前沿专题

波束散射理论和应用

第一讲：小粒子光散射基础

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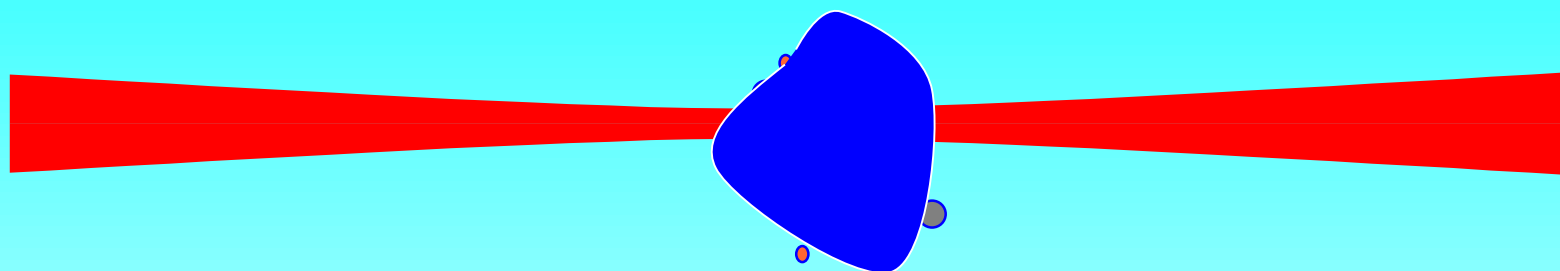
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Plan of lecture

- **Introduction**
- **Fundamentals**
- **Maxwell equations and wave equations**
 - Scalar and vector wave functions
- **Solutions of wave equations**

Introduction



➤ Elastic scattering

➤ Inelastic scattering

➤ Single scattering

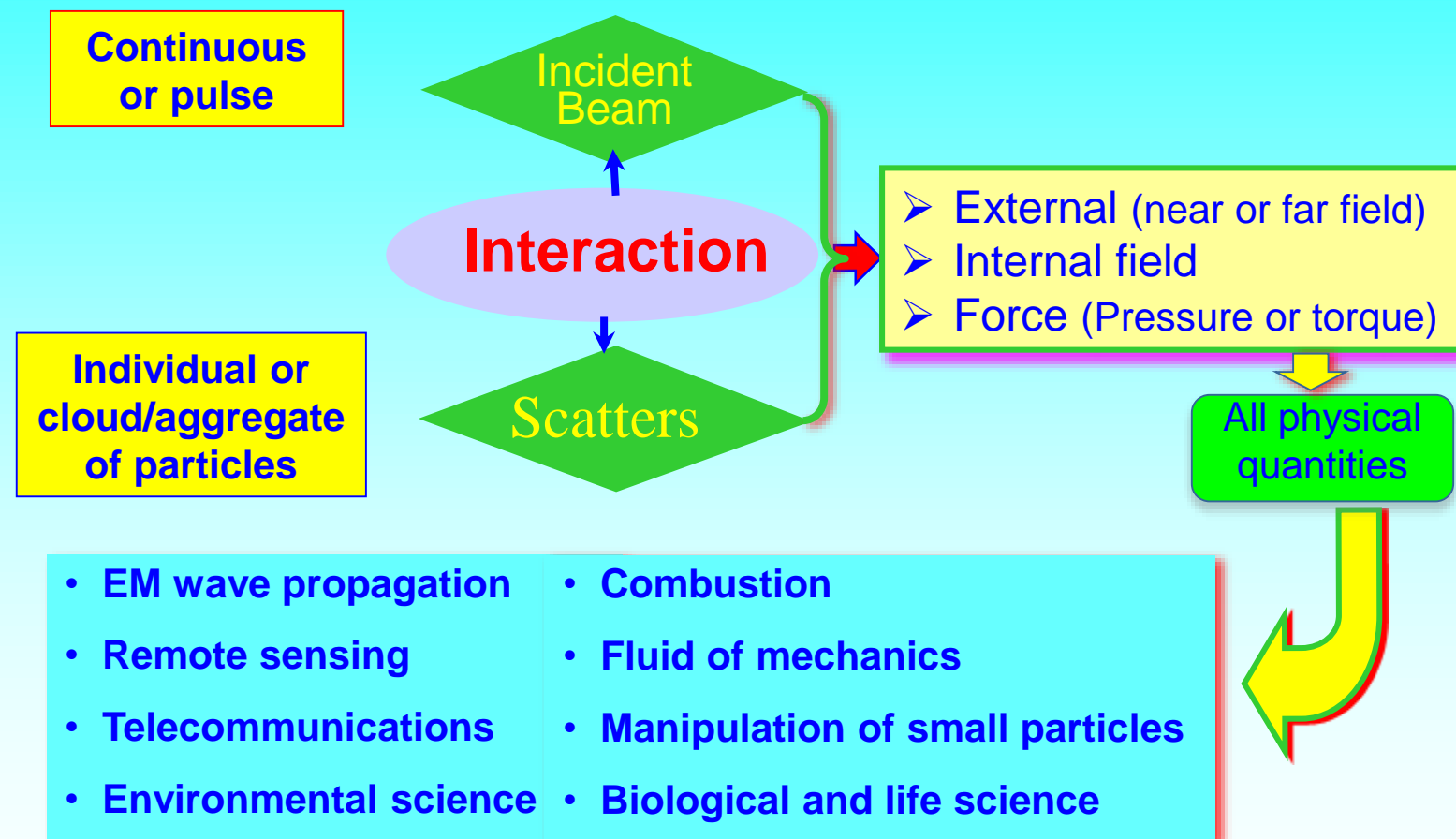
➤ Multiple scattering

➤ Coherent scattering



The effects of multiple scattering and coherent scattering depend not only on the concentration but also the size of the particles.

Introduction



Introduction

Theoretical models

- **Rigorous theories**
 - Lorenz-Mie Theory
 - **Generalized Lorenz-Mie theory (GLMT)**
 - . . .
- **Numerical methods (mainly for non-spherical object)**
 - FDTD - Finite Difference Time Domain
 - MoM - Method of Moments
 - FEM – Finite Element Method
 - T-Matrix
 - DDA - Dipole Discrete Approximation (ADDA and DDSCAT)
 - . . .

Introduction

Theoretical models

➤ Approximate models

- Rayleigh's theory : any shape, dimension $l \ll \lambda$
- Rayleigh-Gans: $|m - 1| \ll 1$
- Diffraction: $l \sim \lambda$
- Geometrical Optics: $l \gg \lambda$
- Geometrical Theory of Diffraction
- Ray theory of wave (RTW) under development
- . . .

Introduction

Applications

Physics

Understanding of physical
procedures

Astrophysics

Effect of force
Interplanetary Dust Particles

EM Scattering of an
arbitrary object

Optical metrology

Environment
Energy

... ..

EM wave

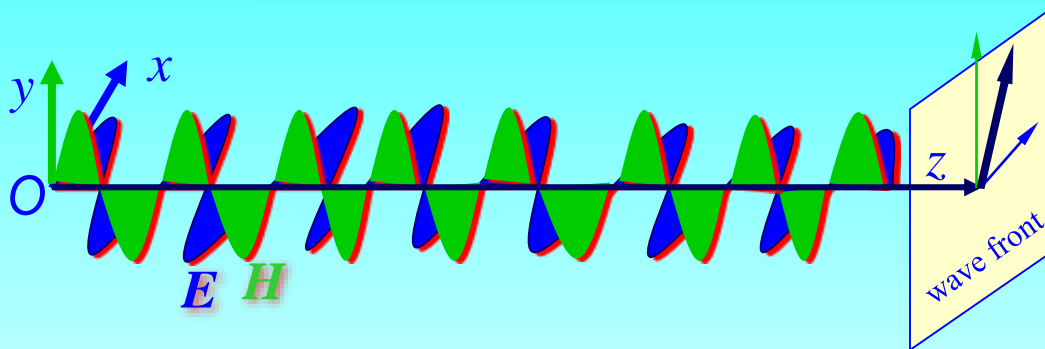
propagation & detection
communication
Radar ...

Optical Tweezers

Biological/life science
Micro-fluidics
Material science

Fundamentals - Plane wave

Electromagnetic (EM) field and properties



Wave front = plane \parallel xOy
 Propagation direction \perp E et H

Wave vector: \mathbf{k}

Wave number: $k = \frac{2\pi}{\lambda}$

Two polarizations:
$$\begin{cases} E_x = A_x \cos(\omega t - \mathbf{k} \cdot \mathbf{r} - \phi_0) \\ E_y = A_y \cos(\omega t - \mathbf{k} \cdot \mathbf{r} - \phi_0) \end{cases}$$

$$\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r} - \phi_0)} \quad (\text{complex fonction})$$

In an isotropic medium:

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H} \quad \mathbf{H} = \frac{1}{\mu \omega} \mathbf{k} \wedge \mathbf{E}$$

\mathbf{E} –electric field

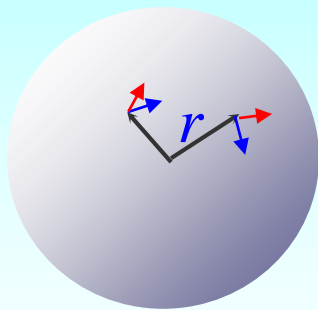
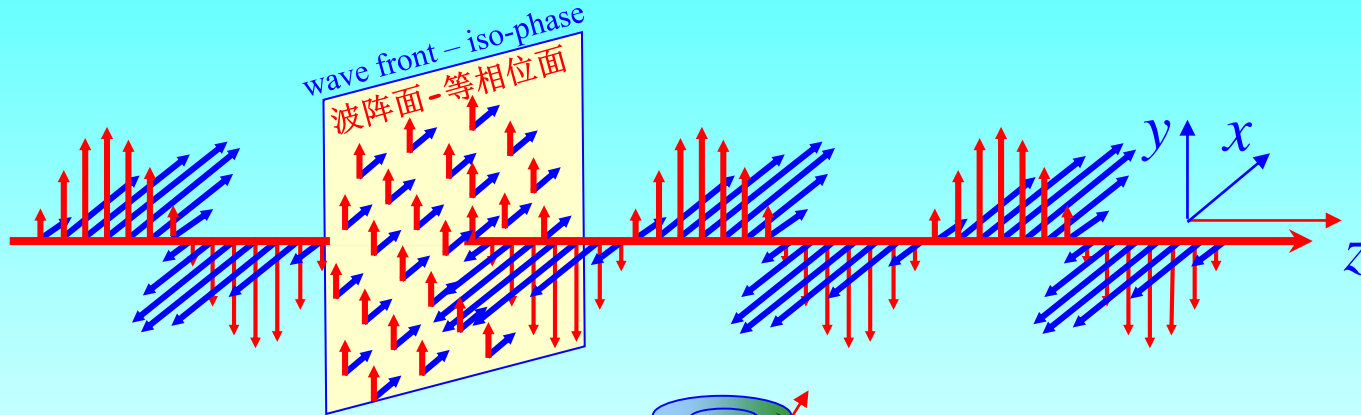
\mathbf{H} –magnetic field

ε – permittivity

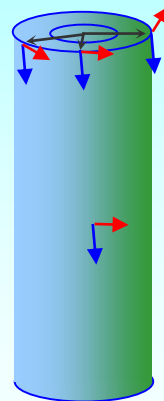
μ – permeability

Fundamentals - Different forms of wave

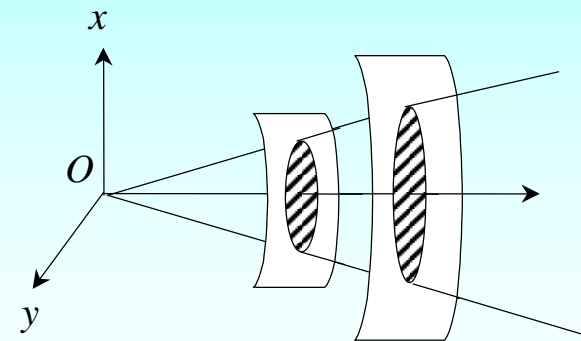
Electromagnetic (EM) field and properties



spherical



Cylindrical



Plane wave in far field

Fundamentals - Refractive index

Complex refractive index

$$\tilde{m} = m_r - m_i i$$

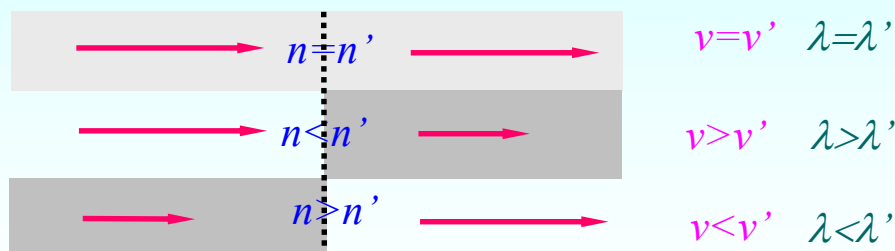
Real part - velocity: $m_r = \frac{c}{v}$

Examples:

vacuum: $c = 3 \times 10^8 \text{ ms}^{-1}$, $\lambda = 0,6328 \text{ }\mu\text{m}$

water: $n_{\text{eau}} = 1,33$ $v_{\text{eau}} = 2,26 \times 10^8 \text{ ms}^{-1}$, $\lambda_{\text{eau}} = 0,4758 \text{ }\mu\text{m}$

glass: $n_{\text{verre}} = 1,5$ $v_{\text{verre}} = 2,00 \times 10^8 \text{ ms}^{-1}$, $\lambda_{\text{verre}} = 0,4219 \text{ }\mu\text{m}$



Fundamentals - Refractive index

Complex refractive index

Imaginary part - **absorption**:

穿透深度:
$$d = \frac{1}{m_i k_0} = 0.16 \frac{\lambda}{m_i}$$

$$\begin{aligned} E &= E_0 e^{i(\omega t - n k_0 z + \phi)} \\ &= E_0 e^{i\omega t - i m_r k_0 z - m_i k_0 z + i\phi} \\ &= E_0 e^{-m_i k_0 z} e^{i(\omega t - m_r k_0 z + \phi)} \end{aligned}$$

Amplitude ? z :

$$E_0(z) = E_0(z=0) e^{-m_i k_0 z}$$

Penetration depth d :

$$\frac{E_0(z=d)}{E_0(z=0)} = e^{-1} \quad \text{i.e.} \quad \frac{I(d)}{I(0)} = \frac{1}{e^2} = 13.5\%$$

$$\Rightarrow d = \frac{1}{m_i k_0} = 0.16 \frac{\lambda}{m_i}$$

$$\lambda = 0.6328 \mu\text{m}$$

$$m_i = 0.1, d = 1 \mu\text{m}$$

$$m_i = 0.0001, d = 1 \text{ mm}$$

Fundamentals - Energy and momentum

Poynting's vector and Intensity

Energy density (J/m^3): $u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$

Poynting's vector (W/m^2):

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*)$$

Complex function



In isotropic medium

Poynting's vector: $\mathbf{S} = v\mathbf{u}\mathbf{n}$

Intensity: $I = \|\mathbf{S}\| \propto E^2$

Fundamentals - Energy and momentum

Stress tensor, force and torque

Stress tensor:

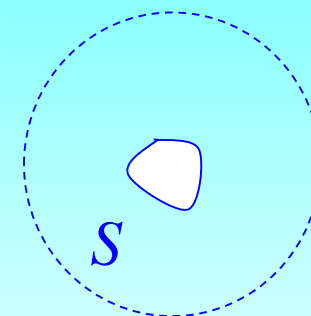
$$\vec{T} = \frac{1}{2} \text{Re} \left[\epsilon \mathbf{E} \mathbf{E}^* + \mu \mathbf{H} \mathbf{H}^* + \frac{1}{2} (\mathbf{E} \cdot \mathbf{E}^* + \mathbf{H} \cdot \mathbf{H}^*) \vec{I} \right]$$

Radiation force:

$$\mathbf{F} = \oint_S d\mathbf{S} \langle \vec{T} \rangle$$

Torque:

$$\mathbf{M} = - \oint_S d\mathbf{S} \cdot (\langle \vec{T} \rangle \times \mathbf{r})$$



Integration over a sphere including the particle:

-when $r \rightarrow \infty$, $E_r \rightarrow 0$:

$$\mathbf{F} = -\frac{1}{4} \int_0^{2\pi} \int_0^\pi \text{Re} \left[\epsilon (|E_\theta|^2 + |E_\phi|^2) + \mu (|H_\theta|^2 + |H_\phi|^2) \right] \mathbf{e}_r r^2 \sin \theta d\theta d\phi$$

-but E_r can never be neglected for torque:

$$\mathbf{M} = -\frac{1}{4} \int_0^{2\pi} \int_0^\pi \text{Re} \left[(\epsilon E_r E_\phi^* + \mu H_r H_\phi^*) \mathbf{e}_\theta - (\epsilon E_r E_\theta^* + \mu H_r H_\theta^*) \mathbf{e}_\phi \right] r^3 \sin \theta d\theta d\phi$$

Fundamentals - Scattering matrix

Relation between incident and scattered waves
in far field

$$\begin{pmatrix} E_{\parallel s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{\parallel i} \\ E_{\perp i} \end{pmatrix}$$

For a particle of spherical symmetry:
 $S_3 = S_4 = 0$

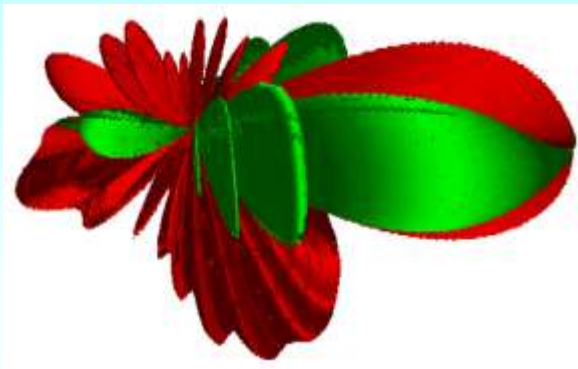
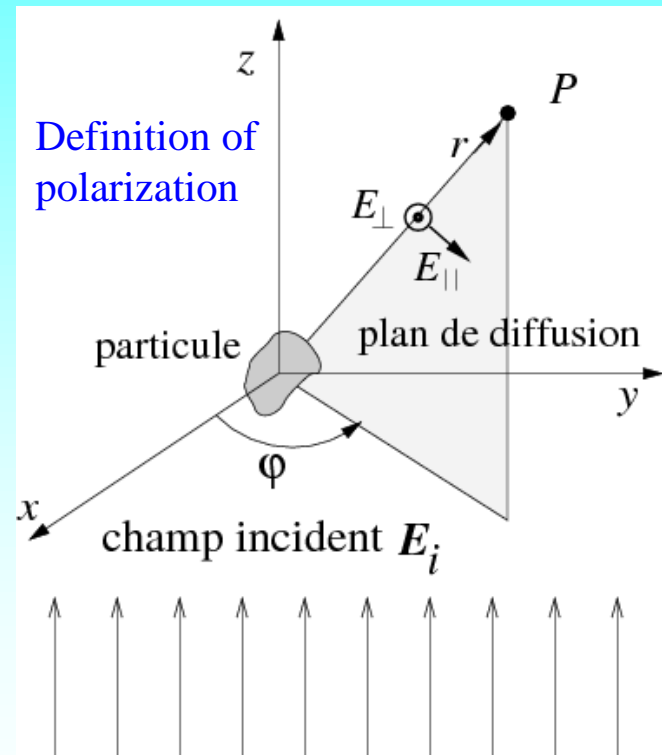


Diagramme de diffusion d'une sphère éclairée par un faisceau gaussien (hors axe), polarisation perpendiculaire en rouge et parallèle en vert.



Definition of polarization

particule

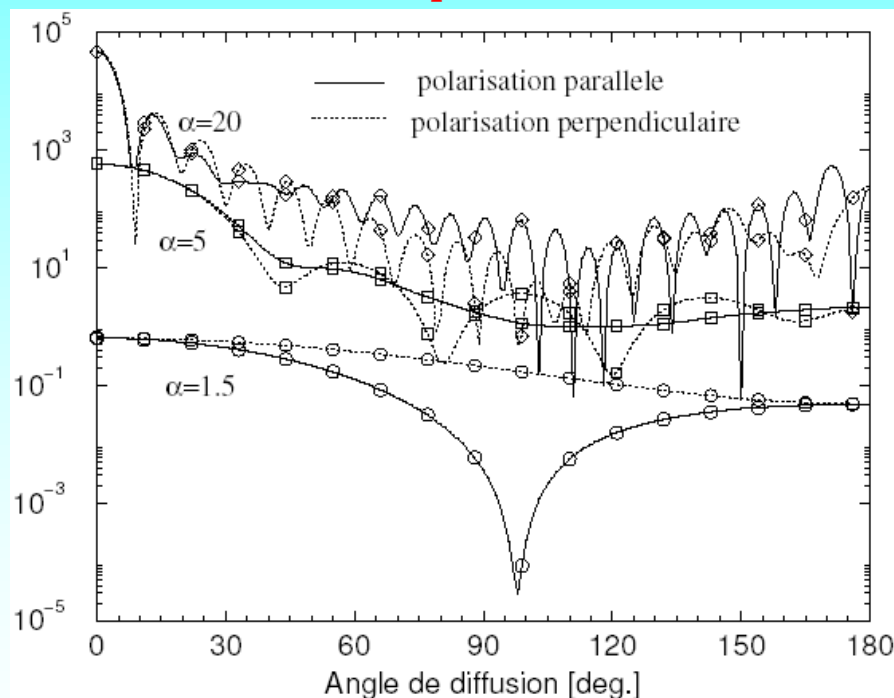
plan de diffusion

champ incident E_i

Fundamentals - Phase function

Scattering diagram

sphere: $d=2a$, refraction index: $m=1.33$
Incident plane wave



$$I(\theta, \varphi) = \frac{I_0 F(\theta, \varphi)}{k^2 r^2}$$

Incident wave polarized
in x direction:

$$I = F(\theta, \phi = 0) = |S_2|^2$$

$$I = F(\theta, \phi = 90^\circ) = |S_1|^2$$

Particle size parameter:

$$\alpha = \frac{\pi d}{\lambda}$$

Fundamentals - Cross sections

Integral properties of a scatter

External EM field:

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s, \quad \mathbf{H} = \mathbf{H}_i + \mathbf{H}_s$$

Poynting vector of total field:

$$\mathbf{S} = \frac{1}{2} \text{Re}\{\mathbf{E} \times \mathbf{H}^*\} = \mathbf{S}_i + \mathbf{S}_s + \mathbf{S}_{ext}$$

$$\mathbf{S}_i = \frac{1}{2} \text{Re}\{\mathbf{E}_i \times \mathbf{H}_i^*\}$$

$$\mathbf{S}_s = \frac{1}{2} \text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$$

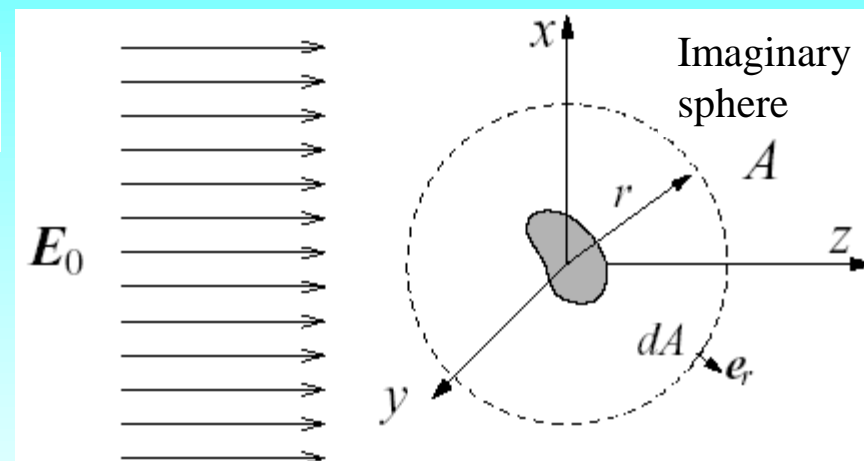
$$\mathbf{S}_{ext} = \frac{1}{2} \text{Re}\{\mathbf{E}_i \times \mathbf{H}_s^* + \mathbf{E}_s \times \mathbf{H}_i^*\}$$

Energy balance

$$W_{abs} = -\int_A \mathbf{S} \cdot \mathbf{e}_r dA = W_{inc} - W_{sca} + W_{ext}$$

$$W_{inc} = -\int_S \mathbf{S}_i \cdot \mathbf{e}_r dA, \quad W_{sca} = -\int_S \mathbf{S}_s \cdot \mathbf{e}_r dA, \quad W_{ext} = -\int_S \mathbf{S}_{ext} \cdot \mathbf{e}_r dA$$

$$W_{ext} = W_{abs} + W_{sca}$$



Fundamentals - Cross sections

Definition of cross sections and efficient factors

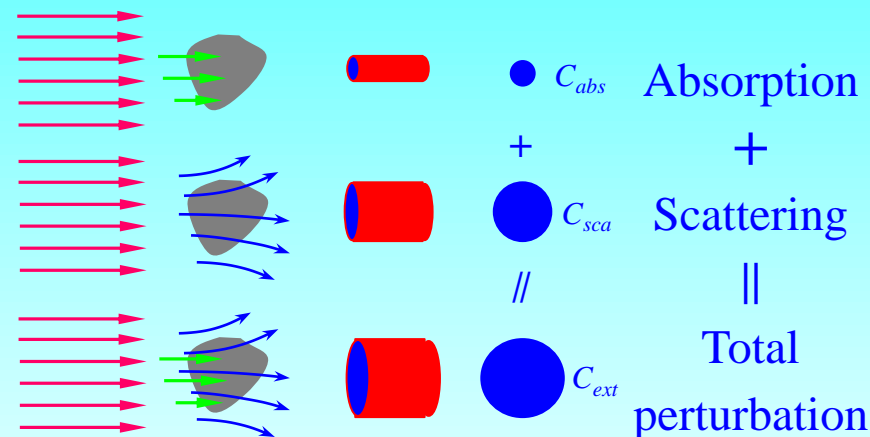
Efficient Sections:

$$\text{Absorption section: } C_{abs} = \frac{W_{abs}}{I_i}$$

$$\text{Scattering section: } C_{sca} = \frac{W_{sca}}{I_i}$$

$$\text{Extinction section: } C_{ext} = \frac{W_{ext}}{I_i}$$

Physical interpretation



Efficiency factors

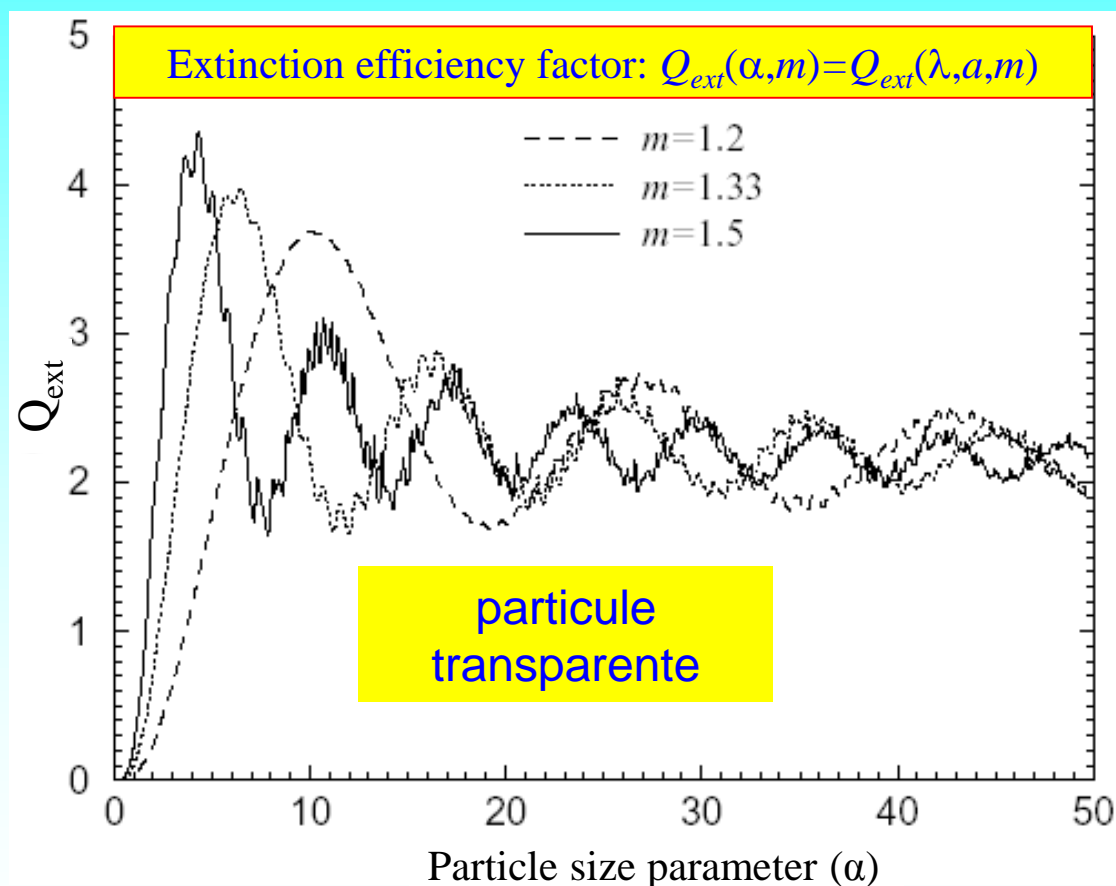
$$Q_{ext} = \frac{C_{ext}}{A}, \quad Q_{abs} = \frac{C_{abs}}{A}, \quad Q_{sca} = \frac{C_{sca}}{A}$$

Transparent particle

$$C_{abs} = 0, \quad C_{ext} = C_{sca} \quad Q_{abs} = 0, \quad Q_{ext} = Q_{sca}$$

Fundamentals - Cross sections

Know to read and use the graph



Small particle

$$Q_{ext} \sim \frac{d^4}{\lambda^4}$$

Large particle

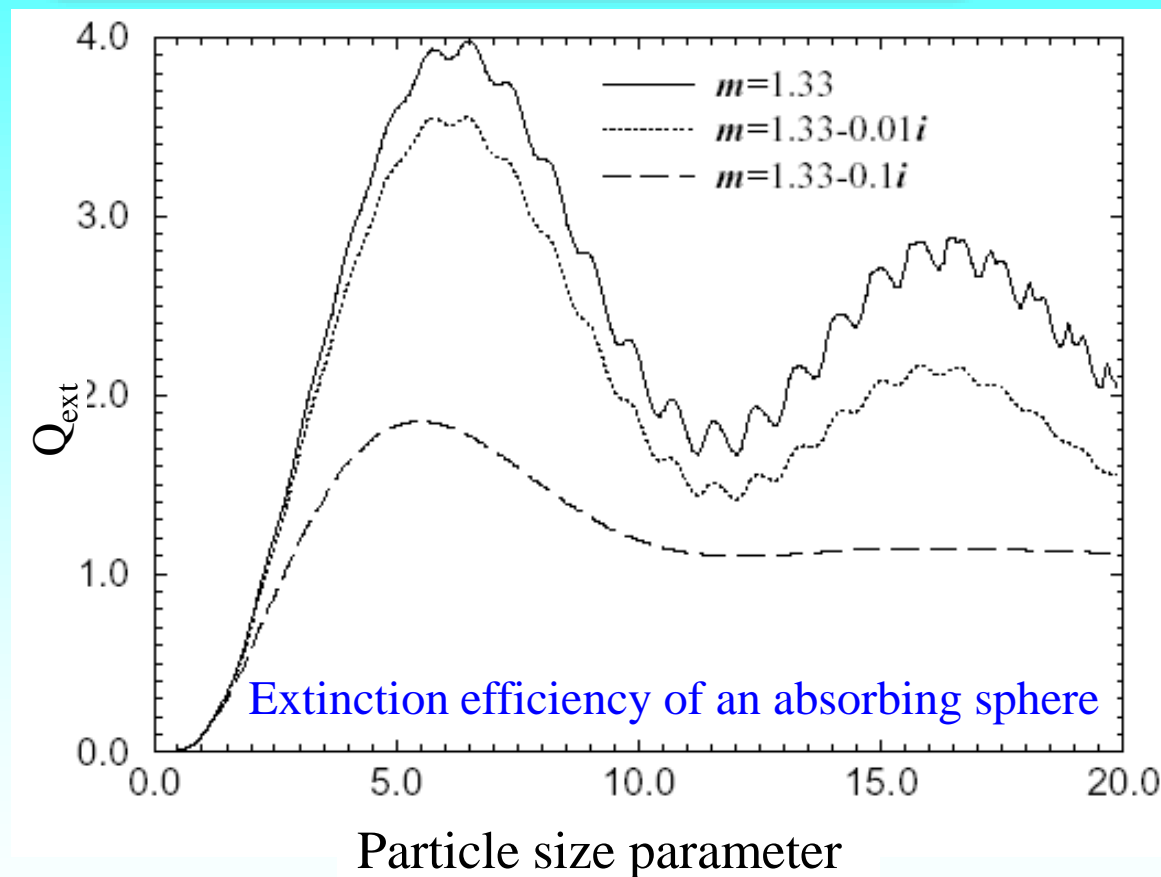
$$Q_{ext} \rightarrow 2$$

The high frequency is smoothed for polydisperse distribution

Why the sky is blue and the sun is red at rising and sunset.

Fundamentals - Cross sections

Know to read and use the graph



Small particle

$d \ll \lambda$:

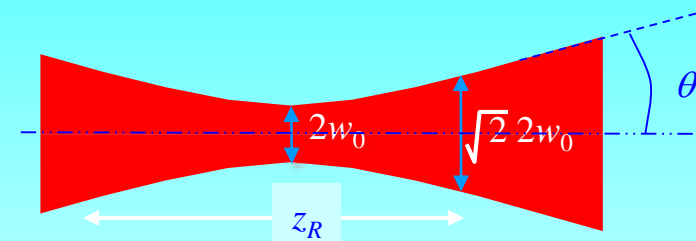
$$Q_{ext} = \frac{8}{3} \left(\frac{\pi d}{\lambda} \right)^4 \operatorname{Re} \left(\frac{m^2 - 1}{m^2 + 2} \right)$$

Fundamentals – Gaussian beam

Characteristics of a beam

(a). Intensity: decreasing along z and r .

$$I(r, z) = I_0 \left[\frac{w_0}{w(z)} \right]^2 \exp \left[-\frac{2r^2}{w^2(z)} \right]$$



(b). Beam waist radius $w_0 = w(z=0)$:

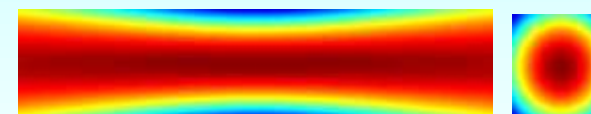
$$I(r=w) = \frac{I(r=0)}{e^2}$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2} \right)^2}$$

$$z_R = \frac{\pi w_0^2}{\lambda}$$

(c). Divergence angle

$$\theta = \lim_{z \rightarrow \infty} \left[\arctan \left(\frac{w(z)}{z} \right) \right] = \arctan \left(\frac{\lambda}{\pi w_0} \right)$$



(d). Rayleigh distance: $z_R = \pi w_0^2 / \lambda$

$$I(0, z_R) = \frac{I_0}{2}, \quad w(z_R) = \sqrt{2} w_0$$

Maxwell equations & wave equations

1. Maxwell equations in differential form

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

\mathbf{E} : electric field
 \mathbf{H} : magnetic field
 \mathbf{D} : electric displacement
 \mathbf{B} : magnetic induction
 \mathbf{j} : electric current density
 ρ : electric charge density

2. Constitutive relations

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H}\end{aligned}$$

ϵ : permittivity
 μ : permeability

ϵ and μ are scalars in an isotropic medium, matrix in an anisotropic medium.

Maxwell equations & wave equations

3. Wave equations of a harmonic wave in free space:

- Harmonic wave: $A(\mathbf{r}, t) = A(\mathbf{r})e^{i\omega t}$ (A stands for H , D , H or B)
- Free space: $\mathbf{j}=0$, $\rho=0$

By calculating the curl of the first two Maxwell equations, using the 3rd and 4th equations and the identity:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

We obtain the wave equations

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0$$

To be checked
by yourself.

with $k^2 = \omega^2 \mu \varepsilon$

- k is the wave number
- ω is the angular frequency
- $v = \sqrt{1/\mu\varepsilon}$ is the wave velocity

Maxwell equations & wave equations

4. Scalar wave equations

- Plane wave: It is evident that the plane wave $E(\mathbf{r}, t) = E_0 e^{i(\omega t \pm \mathbf{k} \cdot \mathbf{r})}$ is a solution of the wave equation.
- General cases: we can show that in free space **two independent scalar functions are sufficient to describe all EM waves**. The two scalar functions can be two components of a vector potential.
- Hertz vectors: We choose often a component of the electric Hertz vector Π_e and a component of the magnetic Hertz vector Π_m as the independent scalar functions and construct the two Hertz vectors:

$$\mathbf{\Pi}_e = \mathbf{a} \Pi_e \quad \text{and} \quad \mathbf{\Pi}_m = \mathbf{a} \Pi_m$$

They satisfy the same wave equation:

$$(\nabla^2 + k^2) \Pi = 0$$

and the EM fields are given by:

$$\begin{aligned} \mathbf{E} &= \nabla \times (\nabla \times \mathbf{\Pi}_e) - i\omega\mu \nabla \times \mathbf{\Pi}_m \\ \mathbf{H} &= i\omega\varepsilon \nabla \times \mathbf{\Pi}_e + \nabla \times (\nabla \times \mathbf{\Pi}_m) \end{aligned}$$

Maxwell equations & wave equations

5. Vector wave equations

- Vector wave equation (A stands for E or H or Π)

$$(\nabla^2 + k^2)A = 0$$

- Vector wave functions (Π stands for either Π_e or Π_m):

We suppose that the scalar function Π satisfies the scalar Helmholtz equation and \mathbf{a} is a vector constant. Then we compose

$$\mathbf{L} = \nabla \Pi$$

$$\mathbf{M} = \nabla \times (\mathbf{a} \Pi)$$

$$\mathbf{N} = \frac{1}{k} \nabla \times \mathbf{M}$$

properties

$$\nabla \times \mathbf{L} = 0$$

$$\nabla \cdot \mathbf{M} = 0$$

$$\nabla \cdot \mathbf{N} = 0$$

$$\nabla \cdot \mathbf{L} = \nabla^2 \Pi = -k^2 \Pi$$

- EM fields:

$$\mathbf{E} = \sum_n (A_n \mathbf{N}_n + B_n \mathbf{M}_n)$$

Cf. Bohren p198

$$\mathbf{H} = \frac{k}{i\omega} \sum_n (A_n \mathbf{M}_n + B_n \mathbf{N}_n)$$

Check this writing from expression of E .

The divergences of \mathbf{E} and \mathbf{H} are null in free space, so no \mathbf{L} .

Solution of wave equations

1. General description

Our task is now to solve the scalar wave equation

$$(\nabla^2 + k^2)\Pi = 0$$

in different coordinate systems by the variable separation method.

i. Wave functions : $\Pi(x_1, x_2, x_3) = X_1(x_1)X_2(x_2)X_3(x_3)$

ii. Differential wave equations:

$$D(\Pi(x_1, x_2, x_3)) = 0$$



$$D(X_1(x_1)) = ?$$

$$D(X_2(x_2)) = \nu$$

$$D(X_3(x_3)) = f(?, \nu)$$

iii. Solutions:

$$\Pi(x_1, x_2, x_3) = \sum_{\mu\nu} C_{\mu\nu} X_1(x_1)X_2(x_2)X_3(x_3)$$

Beam shape coeff.
Scattering coeff.
Internal field coeff.

Usually one or two
special functions

Solution of wave equations

2. Solution in the cylindrical coordinate system

- Differential equations in the spherical coordinate system

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Pi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Pi}{\partial \phi^2} + \frac{\partial^2 \Pi}{\partial z^2} + k^2 \Pi = 0$$

- Separation of the variables. We suppose:

$$\Pi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$$

and obtain:

$$\frac{d^2 Z}{dz^2} + h^2 Z = 0$$

$$\frac{d^2 \Phi}{d\phi^2} + \nu^2 \Phi = 0$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(\mu^2 - \frac{\nu^2}{\rho^2} \right) R = 0$$

Harmonic oscillator differential equations.

Bessel differential eq.

with $h^2 + \mu^2 = k^2$

Solution of wave equations

- General solutions of these three differential equations are respectively

- Exponential function: $\Phi_m(\phi) = e^{im\phi}$ m is the azimuth mode.

- Cylindrical Bessel function: $R_m(\rho) = J_m(\mu\rho)$

- Exponential function: $Z_h(z) = e^{-ihz}$ h is k_z .

So the general solution is given by

$$\Pi_{mh} = Z_m(\mu\rho) e^{i(m\phi - hz)}$$

The Hertz potential for plane wave (h const.) is given by

$$\Pi = \sum_{m=-\infty}^{\infty} c_m Z_m(\mu\rho) e^{i(m\phi - hz)}$$

For shaped beam:

$$\Pi = \sum_{m=-\infty}^{\infty} \int_h c_{mh} Z_m(\mu\rho) e^{i(m\phi - hz)} dh$$

Different fields are expressed with different coefficients and adequate Bessel function.

Solution of wave equations

▪ Electromagnetic field

With help of the relation between the Hertz vectors and EM fields:

$$\mathbf{E} = \nabla \times (\nabla \times \Pi_e) - i\omega\mu \nabla \times \Pi_m$$

$$\mathbf{H} = i\omega\varepsilon \nabla \times \Pi_m + \nabla \times (\nabla \times \Pi_m)$$

We choose $\Pi = \Pi_e \mathbf{e}_z$ and establish

$$E_\rho = \frac{\partial^2 \Pi_e}{\partial \rho \partial z} - \frac{i\omega\mu}{\rho} \frac{\partial \Pi_m}{\partial \phi}$$

$$E_\phi = \frac{1}{\rho} \frac{\partial^2 \Pi_e}{\partial \phi \partial z} + i\omega\mu \frac{\partial \Pi_m}{\partial \rho}$$

$$E_z = \frac{\partial^2 \Pi_e}{\partial z^2} + k^2 \Pi_e$$

$$H_\rho = \frac{\partial^2 \Pi_m}{\partial \rho \partial z} + \frac{i\omega\varepsilon}{\rho} \frac{\partial \Pi_e}{\partial \phi}$$

$$E_\phi = \frac{1}{\rho} \frac{\partial^2 \Pi_m}{\partial \phi \partial z} - i\omega\varepsilon \frac{\partial \Pi_e}{\partial \rho}$$

$$E_z = \frac{\partial^2 \Pi_m}{\partial z^2} + k^2 \Pi_m$$

by using:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \mathbf{e}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \mathbf{e}_\phi + \frac{1}{\rho} \left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \mathbf{e}_z$$

To be checked
by yourself.

Solution of wave equations

- Vector wave functions:

By introducing the Hertz function in the above equations, the EM fields can be expressed as vector wave function in the form like:

$$\mathbf{m} = \nabla \times (\Pi_z \mathbf{e}_z)$$

$$\mathbf{E} = \sum_{m=-\infty}^{\infty} (A_m \mathbf{m}_m + B_m \mathbf{n}_m)$$

with

$$\mathbf{m}_{mh} = \left[\frac{im}{\rho} Z_m(\mu\rho) \mathbf{e}_\rho - \frac{\partial Z_m(\mu\rho)}{\partial \rho} \mathbf{e}_\phi \right] e^{i(m\phi - hz)}$$

$$\mathbf{n}_{mh} = \frac{1}{k} \left[-ih \frac{\partial Z_m(\mu\rho)}{\partial \rho} \mathbf{e}_\rho - \frac{hm}{\rho} Z_m(\mu\rho) \mathbf{e}_\phi + \mu^2 Z_m(\mu\rho) \mathbf{e}_z \right] e^{i(m\phi - hz)}$$

- In classical Lorentz Mie theory – scattering of a plane wave by a cylindrical particle, for a given incident angle ζ , $h = \cos \zeta$ is constant. Only the summation on m is necessary. *But in beam scattering the incident wave and scattered wave **must be expanded** in h , so a integral on h is necessary.*

Solution of wave equations

3. Solution in the spherical coordinate system

- Differential equations in the spherical coordinate system

For convenience we note $\Pi=r\varphi$, then the wave equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2} + k^2 \varphi = 0$$

- Separation of the variables. We suppose:

$$\varphi(r, \theta, \phi) = R(kr)\Theta(\theta)\Phi(\phi)$$

and obtain: $x = \cos \theta, \quad \nu = n(n+1)$

$$\begin{aligned} \frac{d\Phi}{d\phi} + m^2\Phi &= 0 \\ (1-x^2) \frac{d^2P}{dx^2} - 2x \frac{dP}{dx} + \left(\nu^2 - \frac{m^2}{1-x^2} \right) P &= 0 \\ \frac{d^2R}{d(kr)^2} - \frac{2}{kr} \frac{dR}{d(kr)} + \left[1 - \frac{n(n+1)}{(kr)^2} \right] R &= 0 \end{aligned}$$

Solution of wave equations

- General solutions of these three differential equations

$$\left[\frac{d\Phi}{d\phi} + m^2\Phi = 0 \right] \left[(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\nu^2 - \frac{m^2}{1-x^2} \right)P = 0 \right] \left[\frac{d^2R}{d(kr)^2} - \frac{2}{kr}\frac{dR}{d(kr)} + \left[1 - \frac{n(n+1)}{(kr)^2} \right]R = 0 \right]$$

Are respectively

- Exponential function: $e^{im\phi}$
- (Associated) Legendre function: $P_n^m(\cos\theta)$
- Spherical Bessel function: $z_n(kr)$

So the general solution is given by

$$\varphi_{nm} = z_n(kr)P_n^m(\cos\theta)e^{im\phi}$$

The Hertz potential for any EM wave is given by

$$\Pi(r, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_{nm} \begin{pmatrix} \psi_n(kr) \\ \chi_n(kr) \\ \xi_n(kr) \end{pmatrix} P_n^{|m|}(\cos\theta)e^{im\phi}$$

$$\begin{aligned} \psi_n(x) &= xj_n(x) \\ \chi_n(x) &= xh_n^{(1)}(x) \\ \xi_n(x) &= xh_n^{(2)}(x) \end{aligned}$$

Different fields are expressed with different coefficients and adequate Bessel function.

Solution of wave equations

▪ Electromagnetic field

With help of the relation between the Hertz vectors and EM fields:

$$\mathbf{E} = \nabla \times (\nabla \times \Pi_e) - i\omega\mu\nabla \times \Pi_m$$

$$\mathbf{H} = i\omega\varepsilon\nabla \times \Pi_m + \nabla \times (\nabla \times \Pi_m)$$

We choose $\Pi = \Pi e_r$ and establish

$$\begin{aligned} E_r &= \frac{\partial^2 \Pi_e}{\partial r^2} + k^2 \Pi_e \\ E_\theta &= \frac{1}{r} \frac{\partial^2 \Pi_e}{\partial r \partial \theta} - \frac{i\omega\mu}{r \sin \theta} \frac{\partial \Pi_m}{\partial \phi} \\ E_\phi &= \frac{1}{r \sin \theta} \frac{\partial^2 \Pi_e}{\partial r \partial \phi} + \frac{i\omega\mu}{r} \frac{\partial \Pi_m}{\partial \theta} \end{aligned}$$

$$\begin{aligned} H_r &= \frac{\partial^2 \Pi_m}{\partial r^2} + k^2 \Pi_m \\ H_\theta &= \frac{i\omega\varepsilon}{r \sin \theta} \frac{\partial \Pi_e}{\partial \phi} + \frac{1}{r} \frac{\partial^2 \Pi_m}{\partial r \partial \theta} \\ H_\phi &= -\frac{i\omega\varepsilon}{r} \frac{\partial \Pi_e}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial^2 \Pi_m}{\partial r \partial \phi} \end{aligned}$$

by using

To be checked by yourself.

$$\begin{aligned} \nabla u &= \frac{\partial u}{\partial r} e_r + \frac{1}{r} \frac{\partial u}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} e_\phi & \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ \nabla \cdot \mathbf{a} &= \frac{1}{r^2} \frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta a_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial a_\phi}{\partial \phi} \\ \nabla \times \mathbf{a} &= \frac{1}{r \sin \theta} \left[\frac{\partial (\sin \theta a_\phi)}{\partial \theta} - \frac{\partial a_\theta}{\partial \phi} \right] e_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial a_r}{\partial \phi} - \frac{\partial (r a_\phi)}{\partial r} \right] e_\theta + \frac{1}{r} \left[\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta} \right] e_\phi \end{aligned}$$

Solution of wave equations

- Vector wave functions:

By introducing the Hertz function in the above equations, the EM fields can be expressed as vector wave function in the form like:

$$\mathbf{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^n (A_{mn} \mathbf{m}_{mn} + B_{mn} \mathbf{n}_{mn})$$

with

$$\begin{aligned} \mathbf{m}_{mn} &= \left[imz_n(kr)\pi_n^{(m)}(\cos\theta)\mathbf{e}_\theta - z_n(kr)\tau_n^{(m)}(\cos\theta)\mathbf{e}_\phi \right] e^{im\phi} \\ \mathbf{n}_{mn} &= \frac{1}{kr} \left[\frac{n(n+1)}{kr} \psi_n(kr) P_n^{(m)}(\cos\theta) \mathbf{e}_r \right. \\ &\quad \left. + \psi'_n(kr)\tau_n^{(m)}(\cos\theta)\mathbf{e}_\theta + im\psi'_n(kr)\pi_n^{(m)}(\cos\theta)\mathbf{e}_\phi \right] e^{im\phi} \end{aligned}$$

- In classical Lorentz Mie theory – scattering of a plane wave by a spherical particle, we have only terms with $m=\pm 1$, so cosine and sine functions as well as Legendre function are used, and the vector wave functions are noted as \mathbf{m}_{o1n} , \mathbf{m}_{e1n} , \mathbf{n}_{o1n} , \mathbf{n}_{e1n} . *But in beam scattering \mathbf{m}_{mn} , \mathbf{n}_{mn} must be used for the solutions to be completed.*